University of Alberta

Iterative Identification and Predictive Control

by

Ratna Bhushan Gopaluni

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

in

Process Control

Department of Department of Chemical and Materials Engineering

Edmonton, Alberta

Fall 2003

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Degree: Doctor of Philosophy

Year this Degree Granted: 2003

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Date: Sept. 10th, 2003

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విద్యారమ్భం కరిష్యామి సిద్ధిర్భవతు మే సదా //

O Goddess Saraswati, salutations to you, the giver of boons, the one who fulfills all desires. I begin my studies. May there always be accomplishments for me.

Saraswati Namastubhyam

కాయేన వాచ మనసేన్ద్రియైర్వా బుద్ధ్యాత్మనా వా ప్రకృతేస్వభావాత్ //
కరోమి యద్యత్ సకలమ్ పరస్మై నారాయణాయేతి సమర్పయామి //

I dedicate to Narayana, the Supreme God, whatever I have done by the body, speech, mind, and the senses or by the character of nature.

Sandhyavandanam

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Iterative Identification and Predictive Control** submitted by Ratna Bhushan Gopaluni in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in *Process Control.*

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Date: 17th August, 2003

To

అమ్మ నాన్నల కు

My parents and grandparents for the well tuned initial conditions

Abstract

Identification and control have so far been treated mainly as two independent exercises. However, several researchers have shown that identified models that are commensurate with the control algorithm can provide better closed loop performance. Clearly, if the true process model is known then there can not be a better model for any purpose whether it be control, prediction or fault detection. All the identification methods developed for compatibility with the control algorithm, assume that the true model is not known and/or is not identifiable accurately. The central theme of this thesis is about identifying models that are commensurate with model predictive controllers. A commensurate model is expected to provide an overall better closed loop performance.

A new metric for model quality based on the prediction horizon of the controller is defined. A number of properties of this metric under both open and closed loop conditions are derived in this thesis. The effect of this new metric on the bias distribution of the identified models is studied in detail. Expressions for optimal input spectra for obtaining the model predictive control relevant models are also presented in this section.

The dual control problem provides the optimal solution to the problem of online model adaptation and control. However, it is a difficult problem to solve. An approximation

of the dual control problem is provided and its extension to model predictive controllers is outlined. With this as motivation, a new iterative algorithm for improving the closed loop performance is developed based on the metric defined earlier. An improvement of the iterative feedback tuning method is also provided in this part of the thesis. Most of these methods are developed for linear systems. For nonlinear systems, a robust backstepping based iterative adaptive algorithm is also developed.

Every identified model must be qualified with a characterization of its quality. Most of the available methods for quantifying model uncertainty are based on asymptotic statistical properties. Using learning theory, finite sample upper bounds on rates of convergence of identified models are derived for prediction error models.

An offshoot of the research done in this thesis are methods for identification of delay dominated recycle systems and identification of multi-rate systems. An industrial application of the multi-rate identification method is also presented.

Acknowledgements

I am grateful to Prof.Sirish L. Shah for his wonderful guidance and support. He was a source of inspiration through the course of my Ph.D. I have a tendency to get distracted (or may be attracted!) by any new and interesting ideas not totally relevant to my thesis. Sirish always ensured that I focus on my thesis. Without his support and encouragement it wouldn't have been possible to finish this thesis. There are a number of qualities that, I think, I tried to emulate from Sirish. Most of all, his embarrassingly humble demeanor, despite having vast knowledge and experience, is something that I would like to emulate. Sirish gave me all the freedom that any graduate student would crave for in deciding on the topics that I want to work on. I am grateful to him for being very patient during many unproductive periods of research (I was doing course work until Dec 1999!).

There are three persons who made a difference to my life in Edmonton - Rohit, Abdul and Arun. Rohit was instrumental in getting me to work on the topic of this thesis. I am extremely thankful to him for the amount of time he spent with me discussing various ideas. In fact, most of the ideas in this thesis are nothing but a pale imitation of the ideas borrowed from Rohit. He lent me almost all his books for use after leaving the university. In spite of his busy schedule at work he used to meet me regularly to discuss controls. His enthusiasm and intuitive prowess are contagious and I think I tried to imitate him in more than one way. This thesis would not have been possible without his help.

Abdul was one of my roommates for almost four years. I cherish the memories of innumerable discussions we had on religion, philosophy and math though I did not agree

with him on many occasions. Arun is another person who helped me settle down without any problems in Edmonton. He was the only person I knew when I arrived in Edmonton in 1997. He was my roommate for over three years. I learnt a number of techniques involved in cooking from him.

Prof. Huang encouraged me with his many expert comments on my work. Some of the ideas in this thesis were originally developed by Prof. Huang and he was extremely helpful in discussing them and giving many useful comments. I express my gratitude to Prof. Huang for all the help that I received from him. In my third year I was fortunate to have worked with Prof. Mizumoto from Kumamoto University, Japan on adaptive nonlinear methods. I learnt a lot from his expertise in Adaptive and Robust control systems. I am thankful to him and to Dr. Shah for providing me an opportunity to work with him. In the final year I was fortunate enough to have worked with Prof. Patwardhan from Indian Institute of Technology, Chennai. We had a number of interesting discussions on various identification related issues. He introduced me to the area of identification using orthonormal basis functions. We also had a number of interesting discussions on MPCs. I express my gratitude to him for the same. In the same year I had an opportunity to work with Mei Hong from University College, London, U.K on some closed loop identification related issues. He gave me a better perspective of the practical issues in these identification methods. I express my gratitude to him. Profs Chen, Marquez, Guay and Forbes taught me all that I know about nonlinear and robust control and optimization. I am thankful to them.

I cherished the time spent with Abdulkadir, Rodgemar Mamon and Alex De Leon in discussing a number of concepts in stochastic processes. Starting from the all pervasive $(\Omega, \mathcal{F}, P)$ to martingales and markov chains (some times hidden!), I had a number of interesting discussions with them. In fact, my association with them has sparked a keen interest in probabilistic methods to solve the process control problems. I am convinced that the future of process control is in being able to use some of the advanced techniques from probability and statistics.

I am also thankful to Hari, Vinay and Prof. Narasimhan for accompanying me to SUB almost everyday and for all the heated arguments we have had. They have all helped me immensely in understanding a lot of things better.

Finally, I would like to thank a number of fellow students in the CPC group - Laksh, Haitao, Dongguang, Kamrun, Ramesh, Sachin, Misha, Huilan, Ashish, Abhishek, Xin, Hari, Raghu, Mike and Shoukat. The computer process control group (CPC) at University of Alberta is a great place to learn everything about process control. With five faculty members in the group, I had the great opportunity to take many courses in various areas of process control. The group had also provided me with the right environment, with all its laboratory facilities and exceptionally brilliant students, to thrive in both theoretical and practical research. I would specially like to thank, my roommates Mike and Xin for all the interesting arguments we have had.

I would also like to thank the department secretaries - AnnMarie, Leanne and Marilee and the system administrators Bob Barton and Jack Gibeau. I must also acknowledge the help that I received from Walter and Richard in arranging experimental labs for the undergraduate students taking the process control course.

I would also like to thank Siva, my brother-in-law, for his moral support towards the end of my term at the university. Thanks are also due to Vidya for agreeing to accompany me in the random path of life.

Every person's life is like a stochastic process. If $(\Omega,\ \mathcal{F},\ P)$ is the measurable space of God, then a person's life is a mapping $X(\omega,t) : \Omega \times \mathbf{R}^+ \rightarrow \mathbb{L}$. The initial $\omega_0 \in \Omega$ that the process $X(\omega,t)$ takes determines the path of the stochastic variable (in this case the life of a person). Obviously, the initial condition determines the path and that is where the role of my parents- Aruna and Patanjali, my siblings - Harini and Pani and my grandparents is very important. I am what "I am" only because of their love, support and help. No words can express my indebtedness to their unflinching support to do "what I like".

Having said all that, there is an interesting question that I always asked myself but never found a logical answer. How do "I" think? or where am "I" getting all the ideas from (if any)? Is it just a manifestation of my past (memory)? But some times your ideas do not reflect anything that you know. Well, to me the only logical explanation lies in the following axiom.

***Axiom**: Every action and thought is dictated by a Supreme Being.*

Being a believer of the above axiom, it is only appropriate for me to thank Him for everything I have achieved so far.

Finally, I would like to apologize for any mistakes that are still present in this thesis. Rohit and Sirish pointed out numerous mistakes and I have corrected them. However, it is likely that there are certain lingering mistakes in the thesis.

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Part I

Introduction

1 Introduction

This chapter places the work done in this thesis in perspective with the rest of the literature by providing motivational arguments. A summary of various chapters of this thesis is also presented.

The philosophy behind this thesis is succinctly summarized in this quote from Ljung (1999)

> *"The real-life actual system is an object of a different kind than our mathematical models. In a sense, there is an impenetrable but transparent screen between our world of mathematical descriptions and the real world. We can look through this window and compare certain aspects of the physical system with its mathematical description, but we can never establish any exact connection between them. The question of nature's susceptibility to mathematical description has some deep philosophical aspects, and in practical terms we have to take a more pragmatic view of models. Our acceptance of models should thus be guided by "usefulness" rather than "truth". Nevertheless, we shall occasionally use a concept of "the true system", defined in terms of*

a mathematical description. Such a fiction is helpful for devising identification methods and understanding their properties. In such contexts we assume that the observed data have been generated according to some well-defined mathematical rules, which of course is an idealization"

It has been a human endeavor from time immemorial to predict future and use those predictions to change the course of their lives. From astrology to astronomy we have always wanted to predict and know the future. Unfortunately, almost always we can not predict the future accurately. In light of this, our aim has been to predict as closely as possible. In order to predict, we have developed numerous techniques and algorithms. These algorithms result in models that can predict the future with reasonable accuracy. The models are generally based on the past knowledge that we gained through experience. System identification is the science of modelling real life systems from observed data (or the past) and prior knowledge.

Throughout our history we have observed and tried to understand many natural phenomena and affect their course to our liking. From what we see and what we feel, it is possible to describe these phenomenon, through the science of logic and mathematics based on certain fundamental axioms. Depending on the level of abstraction used in describing a phenomenon, we get models with various levels of detail. At the highest level we have what are called phenomological models that are built on certain observed and verifiable scientific facts. The variables involved in phenomological models can either be physically observed (measured) or known to exist. In other words, there seems to be a direct relation between what we see and the *reality*. As a result, this type of modelling seems to be the most natural approach.

However, in practice there are many phenomena that are too complicated to comprehend from a scientific perspective and obtain phenomological models. Many a times, even if such a model is obtained, it is unwieldy for practical purposes. More so, in the process industry. A number of processes in the chemical industry are poorly understood and/or too complicated to model from first principles. Hence, system identification methods based on available data from the process are popular in the industry. These methods are practical, easy to understand and often provide quick solutions. The main disadvantage with these models is that the identified model parameters do not have any relation with the measured physical variables (in many cases) or the relation is too complicated and unknown. System identification methods can provide both linear and nonlinear models. In this thesis, we restrict ourselves to linear models. The area of linear system identification is well developed (Ljung 1999, Söderström and Stoica 1989). In the rest of this thesis we use the words

"process" and "system" interchangeably and call all the variables under our control - inputs. All the measured variables are called outputs.

If the true system is linear and the data is deterministic *i.e.,* if repeating an experiment on the system (also called process in this section) leads to the same output every time, then it is straightforward to identify the process parameters. Unfortunately, many real processes are stochastic *i.e.,* the process exhibits different behavior each time an experiment is conducted even if all the experimental factors remain same. The stochastic nature of the processes makes the identification problem challenging. The idea behind data based identification methods is to lump all the unknown deterministic and stochastic disturbances together and try explaining their impact on the output through a noise model. In this thesis we take a measure theoretic probability approach to describe the noise. We assume that there is an underlying probability space over which the noise term is defined.

An identified model is useful to undertake various tasks. It can be used for control, fault detection and isolation, or merely for prediction. For any purpose, the best identified model is the true process itself if it is available or if it can be identified. It is rarely possible to accurately identify a true process as we do not know the structure or the order of a true process. In the process industry, the objective is to predict the behavior of the processes by modelling and use the model for control, fault detection and isolation, prediction etc. In the current thesis our focus is on control using model predictive controllers.

Model Predictive Controllers are ubiquitous in the process industry. As the name suggests they make use of a model to predict the future behavior of the process given the future inputs. Model Predictive Controllers minimize the predicted control errors at each sample instant to obtain an optimal input trajectory and implement a receding horizon policy by recalculating the optimal input trajectory at each sample instant. In this thesis we explore the connections between model predictive control and identification.

Clearly, since MPC is a function of the process model, the closed loop performance of the system also depends on the model. If the model is known accurately, then MPC can be expected to provide the best performance under closed loop for the objective function considered. In other words, MPC is the optimal controller for the model and not the true process. Inevitably, there is always some amount of uncertainty in the estimated model. The core of this thesis is to shape this uncertainty in such a way that the closed loop performance is satisfactory. The area of robust control deals with designing controllers for uncertain systems.

System identification and robust control theory have occupied the center stage of control research in the last two decades. Only in the late eighties, it was realized that the requirements of robust control theory are not met by the existing system identification techniques (see (Ljung 1999) and (Söderström and Stoica 1989)). It is to fill this wide gap that has developed between the two fields, researchers have come up with the novel idea of joint identification and control. There are three schools of thought pursuing the same goal of marrying system identification with robust control theory (Gevers 1993). They essentially do

1. Estimation of uncertainty bounds on identified models;
2. Identification for robust control design;
3. The combined (synergistic) design of identification and control strategies.

For the moment, the mainstream approach seems to be that of using identification methods which can provide hard or soft uncertainty bounds on the nominal model. As a result, the main focus of research is on the estimation of error bounds (Wahlberg and Ljung 1992, Goodwin *et al.* 1992, Cooley and Lee 1998). These methods can not guarantee that the nominal model provided by them is the best for the particular control design method that is being used. That is, the frequency distribution of the model error may be such that they lead to poor closed loop performance (Schrama 1992*b*, Schrama 1992*a*).

The second line of approach relies on shaping the frequency distribution of the model error in such a way that it gives "optimal" robust performance for the control design method that is being used. Thus, these models will have low uncertainty where closed loop control requires this, and they may have high uncertainty at frequencies where it does not harm the closed loop control performance. Most of these methods perform identification in closed loop with an appropriate data filter. One of the earliest works on closed loop identification was done by Gevers and Ljung (Gevers and Ljung 1986). The book by Bitmead *et al.* (1990) also prescribed closed loop identification with data filters appropriate for robust control. A number of other researchers have also come to the conclusion that control relevant identification must be done under closed loop for better closed loop performance (Schrama 1991).

The third line of approach is to integrate identification and control design into a mutually supportive scheme of identification techniques that lead to robust control. Most of these schemes invariably take the form of an iterative design method involving a succession of identification and robust control steps. A number of such techniques which differ in the identification and robust control criteria have been proposed (Bayard *et al.* 1992, Liu

and Skelton 1990, Partanen and Bitmead 1993, Zang *et al.* 1991, Zang *et al.* 1992).

The idea of retuning the controller based on the closed loop data is very common. In fact that is how controllers are retuned by operators in industry. This research envisages a systematic and theoretical justification for using online plant data to update the controller and the model and thus ensure a high closed loop performance. Our goal is to pursue this third idea of synergistic identification and control. Typically, this approach involves three steps

- Design of Experiments
- Control Relevant Identification and
- Iterative Identification and Control

A block diagram depicting the central idea of this research is shown in fig.1.1. In

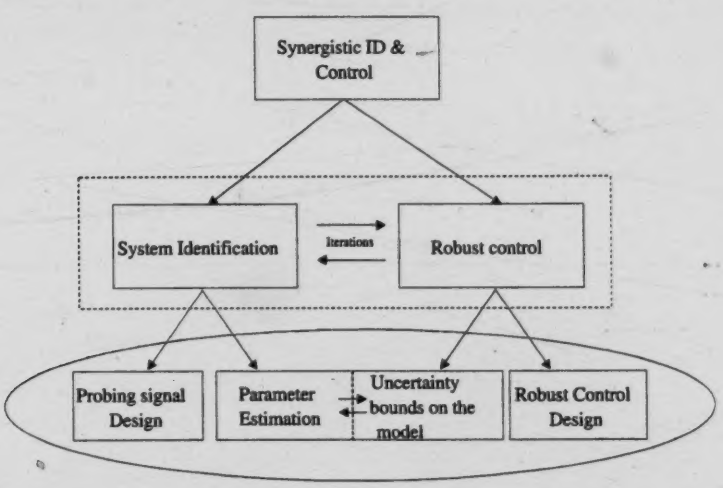


Figure 1.1: Components of a Synergistic Identification and Control system

this thesis, most of these issues are dealt with in the context of a newly defined MPC relevant identification objective function. A summary of various chapters mentioning the important results in each chapter is presented in the next section.

1.1 Outline of this thesis

The central idea in the thesis is a new metric that measures the quality of an identified model. This metric minimizes multi-step ahead prediction errors unlike the traditional identification methods, which minimize one-step ahead prediction errors. This method

is analyzed and developed into an iterative identification and control algorithm for model predictive controllers. Most of the iterative identification and control algorithms available in the literature are some approximation of the dual control problem. Hence, we also present an approximation of the dual control problem. Apart from this, we also present some results on finite sample properties of identified models using learning theory.

This thesis is divided into five main parts. Part I introduces the theme of this thesis briefly. In Part II, a new metric for identifying good multi-step ahead predictors is provided and its properties are studied in detail. We also derive expressions for optimal input spectra in this part of the thesis. In part III, the identification method developed in Part II is developed into an algorithm to perform iterative identification and control. Finite sample properties of the estimated models are studied using learning theory in part IV. A few miscellaneous results obtained while conducting research on the main topic of this thesis are presented in part V. Some of these results also appear as appendices in certain chapters.

1.1.1 MPC relevant identification - Tuning the noise model

In this chapter we introduce the model predictive control relevant identification (called MRI hereafter) by providing certain motivational arguments from the point of view of achieving good closed loop performance. The emphasis in this chapter is on finite impulse response models and therefore for MRI noise models play an important role, hence the title. A number of examples showing the effect of prediction horizon on the identification algorithm are also presented.

1.1.2 Bias distribution in MPC relevant identification

We show that the MRI is useful only when the class of models considered are undermodelled and therefore do not include the true process. A new proof showing that the MRI in fact results in better closed loop performance is presented. The bias distribution of the identified model depends on some function of the noise model and the prediction horizon. We prove a result showing that all the noise models with positive impulse response coefficients improve the bias weighting on the process model in the low frequency region. We also discuss how the bias in this type of identification is effected by various closed loop identification methods.

1.1.3 Iterative Identification and Control

By combining MRI with the model predictive controller, we show that iterative identification and control is possible. The closed loop performance objective function is divided into two parts - one that depends on the identification objective function and the other depends on the control objective function. By iteratively performing identification and control we show that an upper bound on the achieved performance objective function is minimized. A few examples are also presented.

1.1.4 Experiment Design for MRI

Experiment design is an important part of any identification exercise. In this chapter we provide certain expressions for optimal input spectra for MRI under both open and closed loop conditions. The optimal input spectrum is again function of the noise model and the prediction horizon.

1.1.5 An MPC based on an approximation of the dual control problem

An approximation of the dual control problem for FIR models is presented in this chapter. A number of interpretations of this approximation are also elucidated. Based on the interpretations of the approximation it has been extended to more general prediction error models. This approximation can then used in model predictive controllers by posing the controller optimization problem as a Linear Matrix Inequality. This approach to model predictive controllers introduces cautious nature of the dual controllers into control objective function.

1.1.6 Model free iterative controller tuning

A new algorithm based on Nelder-Meade optimization method to tune controllers without identifying a model is developed in this section. We show that popular methods like Iterative Feedback Tuning (IFT) may have slow convergence properties especially when the signal to noise ratio is small. A few examples are presented to show that the proposed derivative free algorithm is at least as effective as the IFT under small signal to noise ratio.

1.1.7 Lyapunov method based nonlinear iterative control

We also look at lyapunov based nonlinear iterative control. Backstepping is one such method. In this chapter a modified backstepping algorithm is applied to a CSTR. The backstepping algorithm is then modified to account for non-affine uncertainty.

1.1.8 Old wine in a new bottle : A learning theoretic approach to system identification

A major portion of this thesis makes use of asymptotic properties of the estimated models. However, in reality our data sets are finite. Hence, it is important to know the rate of convergence of these asymptotic properties. Learning theory is a relatively new area that deals with finite sample properties of measurable functions. We present few results on rate of convergence of the prediction error objective function.

2 Preliminaries

Certain useful results from system identification are presented in this chapter. These results are used in the following chapters. Most of the material presented here can be found in Ljung (1999) and Söderström and Stoica (1989).

2.1 Useful definitions and theorems

The definitions and notations in this thesis follow those in the book by Ljung (1999). A few important definitions and theorems are reproduced here for the sake of completeness.

Stability and uniform s tability

A transfer function, $G(q)$, is said to be stable if

$$G(q) = \sum_{k=1}^{\infty} g(k)q^{-k} \quad \text{and} \quad \sum_{k=1}^{\infty} |g(k)| < \infty \tag{2.1}$$

and is said to be strictly stable if

$$\sum_{k=1}^{\infty} kg(k) < \infty \tag{2.2}$$

Notice that for a transfer function that is rational in q, stability implies strict stability and *vice versa*. Occasionally, it is also necessary to deal with a family of transfer functions, $G(q,\theta)$ where $\theta \in D_{\mathcal{M}}$. It is important to note that the index set $D_{\mathcal{M}}$ need not be finite or countable. A family of transfer functions, $G(q,\theta)$, are said to be uniformly stable if

$$|g_\theta(k)| \leq g(k) \quad \forall\, k \quad \text{and} \quad \sum_{k=1}^{\infty} |g(k)| < \infty \tag{2.3}$$

where $g_\theta(k)$ are the impulse response coefficients of $G(q,\theta)$.

A common framework for deterministic and stochastic signals

In this thesis, a typical process model consists of two parts - a deterministic part that depends only on the input and a stochastic part that depends on the unknown disturbances. It is assumed that the mean of these disturbances is zero. Therefore, for a typical model

$$E[y(t)] \;=\; Gu(t) \tag{2.4}$$

Clearly, the expected value of $y(t)$ is time varying and therefore it is not a stationary signal. Such signals are frequently encountered in identification problems. A common framework for evaluating the second order properties of such signals was proposed in Ljung (1999). In order to deal with mean varying signals, the following definitions are presented.

Definition 2.1.1 (Quasi-stationarity) *A stochastic signal, $s(t)$, satisfying the following properties is said to be quasi-stationary*

(i) $E[s(t)] = m_s(t) \quad |m_s(t)| \leq C \quad \forall\, t$

(ii) $E[s(t)s(r)] = R_s(t,r) \quad |R_s(t,r)| \leq C$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} R_s(t, t-\tau) = R_s(\tau) \quad \forall\, t$$

where the expectation operator is used with respect to the stochastic components of $s(t)$. If $s(t)$ is a stationary stochastic process, the above properties are trivially satisfied.

Every stationary process is also quasi-stationary. In a similar fashion two stochastic processes $\{s(t)\}$ and $\{w(t)\}$ are said to be *jointly quasi-stationary* if both of them are quasi-stationary and the cross covariance function exists *i.e.*,

$$R_{sw}(\tau) = \bar{E}s(t)w(t-\tau)$$

Definition 2.1.2 (Spectra) *For a given quasi-stationary stochastic process, $s(t)$, the spectrum is defined as*

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)e^{-i\omega}$$

and the cross spectrum *between two processes is defined as*

$$\Phi_{sw} = \sum_{\tau=-\infty}^{\infty} R_{sw}(\tau)e^{-i\omega}$$

The following two theorems from Ljung (1999) are extensively used through out this thesis. They are reproduced here for the sake of completeness with alternate and simpler proofs.

Theorem 2.1.1 (Ljung (1999)) *Let $w(t)$ be a quasi-stationary signal with spectrum $\Phi_w(\omega)$, and let $G(q)$ be a stable transfer function. Let*

$$s(t) = G(q)w(t)$$

Then $s(t)$ is also quasi-stationary and

$$\Phi_s(\omega) = |G(e^{i\omega})|^2\Phi_w(\omega) \tag{2.5}$$

Proof: In the book by Ljung (1999) the proof for this theorem was given for a multivariable process. Here, an alternate proof is given for single input single output processes. In order to prove the above result it is necessary to show that the filtered signal, $s(t)$ is quasi-stationary. From the definition of quasi-stationarity it is easy to see

that the signal is quasi-stationary if and only if its covariance function exists. Now

$$\begin{aligned} R_s(\tau) &= \bar{E}[s(t)s(t-\tau)] \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} g(k)g(l)\bar{E}[w(t-k)w(t-\tau-l)] \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} g(k)g(l)R_w(\tau+l-k) \qquad (2.6) \\ &\leq C\left[\sum_{k=1}^{\infty}|g(k)|\right]\left[\sum_{l=1}^{\infty}|g(l)|\right] \\ &< \infty \end{aligned}$$

The last inequality follows due to the stability of the transfer function, $G(q)$. Since, $R_s(\tau)$ is finite, the summation in (2.6) must converge and therefore $s(t)$ is quasi-stationary. The rest of the proof follows the line of argument in Ljung (1999).

□

Corollary 2.1.1 (Ljung (1999)) *Let $y(t)$ be given by*

$$y(t) = G(q)u(t) + H(q)e(t)$$

where $u(t)$ is a quasi-stationary, deterministic signal with the spectrum $\Phi_u(\omega)$, and $e(t)$ is white noise with variance σ_e^2. Let $G(q)$ and $H(q)$ be stable filters. Then $\{y(t)\}$ is quasi-stationary and

$$\Phi_y(\omega) = |G(e^{i\omega})|^2\Phi_u(\omega) + |H(e^{i\omega})|^2\sigma_e^2$$

There is an important practical issue related to the definitions of quasi-stationary signals and their spectra. Their definitions are based on limits of certain expected values. In any realistic situation it is not possible to evaluate these expected values. Instead, only an estimate of them can be obtained from the data. In such a situation, it is important to know if the estimated expected values converge to their true values. This property is known in statistical learning theory as Uniform Convergence of Empirical Means (UCEM). More details on this property and other useful concepts for obtaining bounds on the rate of convergence of estimated means to their true values are presented in Chapter 10. For use in the first few chapters, the following theorem from ergodic theory is presented. The version presented here is equivalent to the one in Ljung (1999).

Theorem 2.1.2 *Let $\{s(t)\}$ be a quasi-stationary signal. Let $Es(t) = m(t)$. Let us suppose that*

$$s(t) - m(t) = \nu(t) = \sum_{k=0}^{\infty} h_t(k)e(t-k) = H_t(q)e(t)$$

where $e(t)$ is a sequence of independent random variables with zero mean values, and variance σ_e^2, and bounded fourth moments, and where $\{H_t(q), t = 1, 2, \cdots\}$ is a uniformly stable family of transfer functions. Then the following limits almost surely converge

$$\begin{aligned} P\left[\frac{1}{N}\sum_{t=1}^{N} s(t)s(t-\tau) \to \bar{E}s(t)s(t-\tau) = R_s(\tau)\right] &= 1 \\ P\left[\frac{1}{N}\sum_{t=1}^{N} s(t)m(t-\tau) \to \bar{E}s(t)m(t-\tau)\right] &= 1 \\ P\left[\frac{1}{N}\sum_{t=1}^{N} s(t)\nu(t-\tau) \to \bar{E}s(t)\nu(t-\tau)\right] &= 1 \end{aligned} \tag{2.7}$$

where the probability measure, P, is the measure on the measurable space over which the stochastic process $e(t)$ is defined.

The above limits indicate that the estimated values of the cross covariance functions converge to their true values everywhere except over a set of measure (with respect to P) zero.

2.1.1 Prediction

Identified models can be used for various purposes. In this thesis, the emphasis is on obtaining good multi-step ahead predictions. Hence, throughout this thesis we make use of optimal predictors extensively. The optimal single step and k-step predictors based on the true model are derived in this section.

In this chapter and the rest of this thesis we assume the following true and estimated process models

$$\mathcal{S} \; : \; y(t) = G(q)u(t) + H(q)e(t) \tag{2.8}$$

$$\hat{\mathcal{S}} \; : \; \hat{y}(t) = \hat{G}(q,\theta)u(t) + \hat{H}(q,\theta)\hat{e}(t) \tag{2.9}$$

where $G(q)$ and $H(q)$ are true the process model and the noise models, $\hat{G}(q,\theta)$ and $\hat{H}(q,\theta)$ are the identified models with θ being the parameter vector. $e(t)$ is the true

normally distributed white noise sequence with zero mean and variance σ_e^2 that enters the process and $\hat{e}(t)$ is fictitious white noise sequence. We assume that the noise model is invertible, *i.e.,* both the noise model and its inverse are stable. This assumption is necessary in order to obtain a stable optimal predictor. If the noise model is not invertible, it is possible that the optimal predictor is also unstable and hence not useful for simulations/predictions. We also assume that the noise model is monic *i.e.,* the first impulse response coefficient is unity. In case the noise model is monic, it is always possible to make it monic by adjusting the noise variance. The optimal one-step ahead predictor is defined as

$$\hat{y}(t+1|t) = E[y(t+1)|\mathcal{F}_t^1] \tag{2.10}$$

where $\mathcal{F}_t^k = \{u(1), \cdots, u(t+k), y(1), \cdots, y(t)\}$, is the σ-algebra (see chapter 10 for definition of σ-algebra) generated by the inputs and outputs. We are interested in finding the "best" possible estimate of the output at a future time instant given all the past and future inputs and the past outputs. Hence,

$$\begin{aligned} \hat{y}(t+1) &= E[y(t+1)|\mathcal{F}_t^1] \\ &= E[Gu(t+1) + He(t+1)|\mathcal{F}_t^1] \\ &= Gu(t+1) + E[(1-H)e(t+1) + e(t+1)|\mathcal{F}_t^1] \end{aligned} \tag{2.11}$$

Now observe that $(1-H)$ has at least unit time delay due to the assumption that H is monic. We can then rewrite the above equation as follows

$$\hat{y}(t+1) = Gu(t+1) + (1-H)e(t+1) \tag{2.12}$$

It is now possible to calculate the values of the past noise, $\{e(s)\}_{s=1}^t$ from (2.8),

$$e(t) = H^{-1}(y(t) - Gu(t)) \tag{2.13}$$

Substituting, (2.13) into (2.12), we get the final expression for optimal one-step ahead predictor

$$\begin{aligned} \hat{y}(t+1) &= Gu(t+1) + (1-H)H^{-1}(y(t+1) - Gu(t+1)) \\ &= H^{-1}Gu(t+1) + (1-H^{-1})y(t+1) \end{aligned} \tag{2.14}$$

This gives us an expression for the optimal predictor if the process model is known. In case the process model is not known, which is often the case, we need to replace the true process and noise models in the above expression with the estimated models *i.e.,*

$$\hat{y}(t+1) = \hat{H}^{-1}\hat{G}u(t+1) + (1-\hat{H}^{-1})y(t+1) \tag{2.15}$$

Subtracting $\hat{y}(t+1)$ from $y(t+1)$, we get what we call the prediction error and is denoted by $\hat{\varepsilon}(t,\theta)$

$$\hat{\varepsilon}(t,\theta) = \hat{H}^{-1}(y(t) - \hat{G}u(t)) \tag{2.16}$$

Having considered the problem of one-step ahead prediction, it is easy to extend it to the following problem : Given all the data up to time instant, t, and the future inputs, we want to obtain the optimal k-step head predictor. In mathematical terms,

$$\hat{y}(t+k|t) = E[y(t+k)|\mathcal{F}_t^k] \tag{2.17}$$

Before we proceed with obtaining an expression for the k-step ahead predictor, let us express the noise model, $H(q)$ in terms of its impulse response coefficients,

$$H(q) := \sum_{i=0}^{\infty} h_i q^{-i} \tag{2.18}$$

Now,

$$\begin{aligned}
\hat{y}(t+k|t) &= E[y(t+k)|\mathcal{F}_t^k] \\
&= E\left[Gu(t+k) + \sum_{i=0}^{\infty} h_i q^{-i} e(t+k) \mid \mathcal{F}_t^k\right] \\
&= Gu(t+k) + \sum_{i=k}^{\infty} h_i q^{-i} e(t+k) + E\left[\sum_{i=0}^{k-1} h_i q^{-i} e(t+k)\right] \\
&= Gu(t+k) + \sum_{i=k}^{\infty} h_i q^{-i} e(t+k)
\end{aligned} \tag{2.19}$$

Let us define,

$$\begin{aligned}
F_k &:= \sum_{i=0}^{k-1} h_i q^{-i} \\
H_k &:= \sum_{i=k}^{\infty} h_i q^{-i}
\end{aligned}$$

Then

$$H := F_k + H_k \tag{2.20}$$

With the above notation, (2.19) can rewritten as

$$\begin{aligned}
\hat{y}(t+k|t) &= Gu(t+k) + \sum_{i=k}^{\infty} h_i q^{-i} e(t+k) \\
&= Gu(t+k) + (H - F_k)e(t+k)
\end{aligned} \tag{2.21}$$

Again, expressing $e(t+k)$ in terms of input-output data, we get

$$\begin{aligned}\hat{y}(t+k|t) &= Gu(t+k) + (H - F_k)H^{-1}(y(t+k) - Gu(t+k)) \\ &= Gu(t+k) + y(t+k) - Gu(t+k) - F_kH^{-1}y(t+k) + F_kH^{-1}Gu(t+k) \\ &= F_kH^{-1}Gu(t+k) + (1 - F_kH^{-1})y(t+k) \end{aligned} \tag{2.22}$$

This expression is extensively used in the rest of this thesis. As in the case of one-step ahead predictions, we can replace all the true models in the above expression with the estimated models to obtain the optimal predictor for the estimated model. We can also evaluate the k-step ahead prediction error,

$$\begin{aligned}\hat{y}(t+k) - y(t+k) &= F_kH^{-1}(y(t+k) - Gu(t+k)) \\ &:= \hat{\varepsilon}(t) \end{aligned} \tag{2.23}$$

2.1.2 Transfer-Function Models

The estimated model is obviously a function of the parametrization used. There are numerous ways of parameterizing the model set. The most general model structure used in the prediction error methods is

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t) \tag{2.24}$$

where

$$\begin{aligned} A(q) &= 1 + a_1q^{-1} + \cdots + a_{n_a}q^{-n_a} \\ B(q) &= b_1q^{-1} + \cdots + b_{n_b}q^{-n_b} \\ C(q) &= 1 + c_1q^{-1} + \cdots + c_{n_c}q^{-n_c} \\ D(q) &= 1 + d_1q^{-1} + \cdots + d_{n_d}q^{-n_d} \\ D(q) &= 1 + f_1q^{-1} + \cdots + f_{n_f}q^{-n_f} \end{aligned}$$

and

$$\theta = [a_1, \cdots, a_{n_a}, b_1, \cdots, b_{n_b}, c_1, \cdots, c_{n_c}, d_1, \cdots, d_{n_d}, f_1, \cdots, f_{n_f}] \tag{2.25}$$

Depending on the form that $A(q), B(q), C(q), D(q), F(q)$ take, the above model structure is given various names. The Table 2.1.2 shows different model structures and what they are called. In this thesis, we mainly deal with Box-Jenkins (BJ) models, which has the most flexible form of parameterizing the model structure. It is important to note that all the models can be converted to ARMAX form with particular choices of the polynomials involved. However, in this thesis, we use the BJ form.

Model	A(q)	B(q)	C(q)	D(q)	F(q)
Output Error	1	B(q)	1	1	F(q)
ARX	A(q)	B(q)	1	1	1
ARMAX	A(q)	B(q)	C(q)	1	1
Box-Jenkins	1	B(q)	C(q)	D(q)	F(q)

Table 2.1: Model Structures

2.1.3 Model sets and Model structures

The following standard definitions are used throughout this thesis. As in Ljung (1999), we distinguish model sets and model structures and define identifiability. Generally, an identification algorithm involves searching for the best model in a class of model sets *i.e.,* searching for the model among the given class that best approximates the data collected from the real process. In order for us to evaluate the quality of an identified model and derive asymptotic properties, it is desirable to have certain properties for the model sets. Hence, in this section we provide some of those definitions. The material in this section is discussed in greater detail in chapter 4 of Ljung (1999).

Definition 2.1.3 (Model Set) *A model set $\mathcal{M}^*$ is defined as*

$$\mathcal{M}^* = \{W_\alpha(q) : \alpha \in \mathcal{A}\} \tag{2.26}$$

where $W_\alpha(q)$ is a discrete transfer function and $\mathcal{A}$ is the index set. Note that this index set need not be continuous, it can as well be a discrete set - for instance, $\mathcal{M}^*$ can just be a collection of few models.

Definition 2.1.4 (Model Structure) *A model structure is a differentiable mapping from a connected open subset $D_\mathcal{M}$ of $\mathbb{R}^d$ to a model set $\mathcal{M}^*$, such that the gradients of the predictor functions are stable. Mathematically,*

$$\mathcal{M} : D_\mathcal{M} \rightarrow \mathcal{M} \tag{2.27}$$

Model structures are a subset of a model set. A model set can be a collection of any set of models. This set can be finite, countable or uncountable depending on the index set. On the other hand, a model structure is a subset of a model set with certain desirable properties. A model structure must be differentiable with respect to the parameters. This property is useful when minimizing the identification objective function and in estimating the asymptotic properties of the identified model.

Once a model structure is chosen for identification, it is important to know if the minimization of an identification objective function leads to a unique "optimal" model. Now this is possible only if each model in the model structure is distinguishable from every other model structure. Distinguishability of model structures depends on the information in the identification data set and the model structure itself. If the data is informative enough to distinguish any two models, then the question is whether two different model parameter sets lead to the same model. In order to answer this question, we define identifiability.

Definition 2.1.5 (Global identifiability at θ^*) *A model structure $\mathcal{M}$ is* ***globally identifiable*** *at θ^* if*

$$\mathcal{M} = \mathcal{M}(\theta^*) \qquad \theta \in D_{\mathcal{M}} \Rightarrow \theta = \theta^*. \tag{2.28}$$

Definition 2.1.6 (Strict global identifiability) *A model structure $\mathcal{M}$ is* ***strictly globally identifiable*** *if it is globally identifiable at all $\theta^* \in D_{\mathcal{M}}$.*

Definition 2.1.7 (Global identifiability) *A model structure $\mathcal{M}$ is* ***globally identifiable*** *if it is globally identifiability at almost all $\theta^* \in D_{\mathcal{M}}$.*

In this thesis, we generally assume that the input excitation is informative enough to distinguish various models in the model structure unless otherwise stated. We only consider model structures that are distinguishable.

2.1.4 Parameter estimation methods

Let us assume that we have selected a model set and an associated model structure within which the true system is likely to lie. The search for the best model then becomes one of finding the best possible parameter vector θ such that the true behavior of the process is approximated as closely as possible to the true system. There are different views on how to determine closeness. In this thesis, we concentrate on a particular method of determining the closeness of the identified model to the true system by defining a metric that minimizes multi-step ahead prediction errors.

Let us denote the data collected by

$$Z^N = [u(1), y(1), \cdots, u(N), y(N)] \tag{2.29}$$

where N is the data length, then the parameter estimation method is a mapping

$$Z^N \rightarrow \hat{\theta}_N \in D_{\mathcal{M}} \tag{2.30}$$

In this thesis, we only concentrate on the what are called prediction error methods. Prediction error methods measure the closeness of the estimated model to the true system by minimizing the prediction errors *i.e.,* an objective function of the following form is defined to obtain the identified model

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N} f[\hat{\varepsilon}(t, \theta, Z^N)] \tag{2.31}$$

where $f(.)$ is a positive function of the prediction errors. In general it can be any norm in $\mathbb{R}^N$. Note the explicit dependence of the objective function on the data and its length. The simplest and the most commonly used metric is

$$f[\hat{\varepsilon}(t, \theta, Z^N)] := \hat{\varepsilon}(t, \theta, Z^N)^2 \tag{2.32}$$

The actual parameter estimate is then defined as

$$\hat{\theta}_N = \underset{\theta \in D_{\mathcal{M}}}{\text{argmin}}\ V_N(\theta, Z^N) \tag{2.33}$$

where "argmin" stands for the minimizing argument of the function. If the minimum is not unique then we let argmin denote the set of minimizing arguments. Some times, depending on the flexibility of the model structure and the specific needs of the user, it may be desirable to filter the prediction errors and then construct an objective function with the filtered prediction errors. In such a case, we have

$$\begin{aligned} \hat{\varepsilon}_F(t, \theta, Z^N) &:= L(q)\hat{\varepsilon}(t, \theta, Z^N) \\ V_N(\theta, Z^N) &= \frac{1}{N}\sum_{t=1}^{N} f[\hat{\varepsilon}_F(t, \theta, Z^N)] \end{aligned} \tag{2.34}$$

where $L(q)$ is a filter of user's choice.

2.1.5 Convergence and Consistency

In the previous section we have outlined some of the often used prediction error methods. For any user to have confidence in the estimated model he needs to be supplied with the quality of that model. There are a number of factors that can be used to quantify the quality of a model. The significant among them are

- Does $\hat{\theta}_N$ converge to, say the true parameter vector, θ^*? If the true model does not lie in the model set chosen for identification, then does $\hat{\theta}_N$ converge to the best model in the set?

- What is the covariance of the estimated parameters and the estimated transfer function?

In order to answer these question, in Ljung (1999), certain assumptions are made. We reproduce, for the sake of completeness, those assumptions and the important lemmas in this section.

A1. The data set Z^∞ is such that for some filters $\{g_t^{(i)}(k)\}$

$$\begin{aligned} y(t) &= \sum_{k=1}^{\infty} g_t^{(1)}(k)w(t-k) + \sum_{k=0}^{\infty} g_t^{(2)}(k)e(t-k) \\ u(t) &= \sum_{k=1}^{\infty} g_t^{(3)}(k)w(t-k) + \sum_{k=0}^{\infty} g_t^{(4)}(k)e(t-k) \end{aligned}$$

where

1. $\{w(t)\}$ is a bounded, deterministic, external input sequence.
2. $\{e(t)\}$ is a sequence of independent random variables with zero mean values and bounded moments of order $4+\delta$ for some $\delta > 0$.
3. The family of filters $\{g_t^{(i)}\}_{k=1}^{\infty}, i = 1,2,3,4; t = 1,2,\cdots$ is uniformly stable.
4. The signals $\{y(t)\}, \{u(t)\}$ are jointly quasi-stationary.

Lemma 2.1.1 *Consider a uniformly stable, linear model structure $\mathcal{M}$. Assume that the data set Z^∞ is subject to A1. Then with $V_N(\theta, Z^N)$ defined by (2.34), we have*

$$P\{\sup_{\theta \in D_{\mathcal{M}}} |V_N(\theta, Z^N) - \bar{V}(\theta)| \to 0\} = 1 \qquad \text{as } N \to \infty \tag{2.35}$$

where

$$\bar{V}(\theta) = \bar{E}\frac{1}{2}\hat{\varepsilon}(t,\theta)^2 \tag{2.36}$$

where P is the probability measure over the measurable space of $e(t)$. The identification objective function thus converges uniformly in $\theta \in D_{\mathcal{M}}$ to the limit function $\bar{V}(\theta)$. This property of the square of prediction errors is known as Uniform Convergence of Empirical Means (UCEM) (see Chapter 10 for details) in statistical learning theory. We deal with some of these concepts from a learning theoretic point of view in Chapter 10.

Theorem 2.1.3 *Suppose that $\hat{\theta}_N$ be defined by* (2.33)*, where $\hat{\varepsilon}(t,\theta)$ is determined from a uniformly stable linear model structure $\mathcal{M}$. Assume that the data set Z^∞ is subject to A1. Then*

$$P\left\{|\hat{\theta}_N - \underset{\theta \in D_{\mathcal{M}}}{\operatorname{argmin}} \bar{V}(\theta)| \to 0\right\} = 1 \qquad \text{as } N \to \infty \tag{2.37}$$

Remark 2.1.1 While the Lemma 2.1.1 ensures convergence of the estimate of the objective function to the true objective function, theorem 2.1.3 ensures convergence of the parameter vector to the best parameter vector as the data length tends to infinity. This also shows that the second order properties of the prediction errors determine the convergence of the parameters. The above results ensure convergence of the estimates but they do not provide any insight into the rate of convergence. It is indeed useful to know the rate of convergence in order estimate the model uncertainty from finite sample data. Chapter 10 attempts to address some of these issues.

In Ljung (1999) a number of other results on identifiability and consistency of uniformly stable structures are also presented. These provide conditions for convergence to the true model if it lies in the model structure considered. For further reading material on the consistency issues the reader is referred to Ljung (1999).

2.1.6 Asymptotic properties of parameter estimates

Once we have understood the convergence properties of the parameter estimate $\hat{\theta}_N$, it is of interest to know the asymptotic distribution of the convergent parameters. In other words, we are interested in the asymptotic distribution of the parameter vector $\hat{\theta}_N$. Observe that $\hat{\theta}_N$ is a stochastic variable due to its dependence on the stochastic input-output data. In this context, the following theorem from Ljung (1999) is useful.

Theorem 2.1.4 *Let us consider the estimate, $\hat{\theta}_N$, obtained by (2.33). Assume the model structure is linear and uniformly stable and that the data setZ^∞ is subject to A1. Assume also that for a unique value θ^*, we have*

$$\left.\begin{aligned} P\{|\hat{\theta}_N - \theta^*| \to 0\} &= 1 \qquad as\ N \to \infty \\ \bar{V}''(\theta) &> 0 \end{aligned}\right\} \tag{2.38}$$

and that

$$\sqrt{N} D_N \to 0, \qquad as\ N \to \infty \tag{2.39}$$

with D_N defined as

$$D_N = E\left(\left[\bar{E}\left(\frac{d}{d\theta}\hat{\varepsilon}(t,\theta)\,|_{\theta=\theta^*}\right)\hat{\varepsilon}(t,\theta^*)\right] - \frac{1}{N}\sum_{t=1}^{N}\left[\left(\frac{d}{d\theta}\hat{\varepsilon}(t,\theta)\,|_{\theta=\theta^*}\right)\hat{\varepsilon}(t,\theta^*)\right]\right). \tag{2.40}$$

Then

$$\sqrt{N}(\hat{\theta} - \theta^*) \in \mathcal{N}(0, P_\theta) \qquad As\ N \to \infty \tag{2.41}$$

where $\mathcal{N}(.,.)$ denotes normal distribution and P_θ is given by

$$P_\theta = [\bar{V}''(\theta^*)]^{-1} \lim_{N\to\infty} .E\left(V_N'(\theta^*, Z^N)V_N'(\theta^*, Z^N)^T\right)[\bar{V}''(\theta^*)]^{-1} \tag{2.42}$$

Remark 2.1.2 The above theorem shows that the estimated parameter vector is normally distributed with mean θ^* and a certain distribution. This expression for the variance of the parameters is useful in evaluating the asymptotic properties of the estimated transfer function. The following expressions for variance of transfer function estimates are useful.

$$Cov\begin{bmatrix} \hat{G}_N(e^{i\omega}) \\ \hat{H}_N(e^{i\omega}) \end{bmatrix} \sim \frac{n}{N}\Phi_\nu(\omega)\begin{bmatrix} \Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_eu(\omega) & \Phi_e(\omega) \end{bmatrix}^{-1} \tag{2.43}$$

where

$$\nu(t) = H(q)e(t) \tag{2.44}$$

Φ_u, Φ_{ue} etc., are corresponding signal spectra, n is the model order. The above expression for variance is valid only as both the model order and the data length tend to infinity. Estimating the transfer function variance for fixed model orders is an open problem and has not yet been solved. Learning theory can perhaps help us in solving this problem too. If the model order is fixed and if the true model does not fit the selected structure, then the prediction errors are not white. This issue complicates the evaluation of variance properties of the estimates.

2.1.7 Recursive identification

Sometimes it may be necessary to identify a model of the process while the system is under operation. The model can then be based on the observations up to the current time instant. The need for such an online model identification algorithm arises due to various considerations. This type of online identification methods are called adaptive. While performing online identification it is important to ensure that the estimation algorithm at each instant is simple enough to compute in one sample instant.

In this section we provide certain expressions for parameter and variance update that are used in Chapter 8. The material presented here is well known in the area of recursive identification and is taken from Ljung (1999). Consider a predictor of the form

$$\hat{y}(t|t-1) = \phi^T(t)\theta + \mu(t) \tag{2.45}$$

where $\hat{y}(t|t-1)$ is the one-step ahead predictor and $\phi(t) = [-y(t-1), -y(t-2), \cdots, -y(t-n_a), u(t-1), \cdots, u(t-n_b)]^T$. $\mu(t)$ is known data-dependent vector. Also

suppose that the following prediction error objective function is used

$$V_t(\theta, Z^t) = \sum_{k=1}^{t} \beta(t,k)[y(k) - \phi^T\theta]^2. \tag{2.46}$$

where $\beta(t,k) = \lambda(t)\beta(t-1,k)$ for $1 \leq k \leq t-1$ and $\beta(t,t) = 1$. $\lambda(t)$ is a positive weighting factor. Then the following expressions for parameter and variance update can easily be obtained (Chapter 11, Ljung (1999)).

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)[y(t) - \phi^T(t)\hat{\theta}(t-1)] \tag{2.47}$$

$$L(t) = \frac{P(t-1)\phi(t)}{\lambda(t) + \phi^T(t)P(t-1)\phi(t)} \tag{2.48}$$

$$P(t) = \frac{1}{\lambda(t)}\left[P(t-1) - \frac{P(t-1)\zeta(t)P(t-1)}{\lambda(t) + \phi^T(t)P(t-1)\phi(t)}\right] \tag{2.49}$$

Similar expressions can be derived for recursive prediction error methods. These expressions are also used in Chapter 7 in the context of approximating the dual control problem. For a general prediction error model in (2.24), define an objective function

$$V_t(\theta, Z^t) = \gamma(t)\frac{1}{2}\sum_{k=1}^{t} \beta(t,k)\hat{\varepsilon}^2(k,\theta,Z^t) \tag{2.50}$$

where

$$\gamma(t) = \left[\sum_{k=1}^{t} \beta(t,k)\right]^{-1}. \tag{2.51}$$

Then it is possible to derive expressions for recursive parameter identification and variance update.

$$\hat{\varepsilon}(t) = y(t) - \hat{y}(t) \tag{2.52}$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \gamma(t)R^{-1}(t)\psi(t)\hat{\varepsilon}(t) \tag{2.53}$$

$$R(t) = R(t-1) + \gamma(t)[\psi(t)\psi^T(t) - R(t-1)] \tag{2.54}$$

It is possible to prove that the asymptotic properties of the recursive prediction error method on a time-invariant system, coincide with those of the off-line prediction error methods presented in an earlier section.

2.1.8 Experiment design

One of the important steps in system identification is experiment design. As we have seen in one of the earlier sections, the identifiability of model depends on the informativeness

of the data collected. The "richer" the data is, the better is the model quality. If we can excite the real process in all frequency ranges for sufficient duration, then it is possible to obtain excellent models. However, in real industrial situations it is generally not possible to have sufficient excitation. Under these restrictions, it is important to design an input spectrum that best extracts the important features of the system. Traditionally, experiment design meant selecting a number of variables, *viz.*, choosing inputs and outputs, sample interval, number of samples and input spectrum. In this thesis, we concern ourselves with only the input spectrum. In this context we present formal definitions of an informative data set and persistence excitation. These definitions are later used in Chapter 5 in the context of experiment design for model predictive control relevant identification.

Definition 2.1.8 (Informative data set)
A quasi-stationary data set Z^∞ is ***informative enough*** *with respect to the model set $\mathcal{M}^*$ if, for any two models $W_1(q)$ and $W_2(q)$ in the set,*

$$\bar{E}\left[(W_1(q) - W_2(q))z(t)\right]^2 \equiv 0 \tag{2.55}$$

where $z(t) = [u(t)\ \ y(t)]$. In other words this implies that $W_1(e^{i\omega}) = W_2(e^{i\omega})$ for almost all ω. Mathematically, the measure of the set of values of ω at which $W_1(e^{i\omega}) \neq W_2(e^{i\omega})$ is zero.

Definition 2.1.9 (Persistent excitation) *A quasi-stationary signal $u(t)$, with spectrum $\Phi_u(\omega)$, is said to be* ***persistently exciting*** *of order n if, for all filters of the form*

$$Q_n(q) \ = \ m_q^{-1} + \cdots + m_n q^{-n} \tag{2.56}$$

the relation

$$|Q_n(e^{i\omega})|^2 \Phi_u(\omega) \equiv 0 \tag{2.57}$$

implies that $Q_n(e^{i\omega}) \equiv 0$.

Remark 2.1.3 It is easy to see that $|Q_n(e^{i\omega})|$ has at most $n-1$ distinct zeros on the unit circle. Hence $u(t)$ is persistently exciting of order n, if $\Phi_u(\omega)$ is different from zero at least at n points in the interval $-\pi < \omega \leq \pi$. We say that $\Phi_u(\omega)$ is **persistently exciting** if

$$\Phi_u(\omega) > 0 \qquad \forall\ \omega \tag{2.58}$$

2.2 Conclusions

In this part of the thesis, we have provided some introductory material that will be used in the remaining chapters. Certain motivational arguments for introducing the concept of model predictive control relevant identification are presented. The work of this thesis has been placed in perspective with the existing literature. A summary of all the chapters in this thesis and their interconnections are also explained.

Part II

Model Predictive Control Relevant Identification

3

MPC Relevant Identification - tuning the noise model

The philosophy of identification by minimizing an objective function that is commensurate with the control objective function is called control relevant identification. The control relevant method studied in this chapter minimizes a multi-step ahead prediction error objective function, suitable for model predictive controllers, to obtain an optimal multi-step ahead predictor. It is shown that the method described in this chapter provides better designed performance of the controller. A number of properties of this method in the context of FIR models are presented in this chapter. The noise model plays a pivotal role in determining the performance of multi-step ahead prediction errors. A method for tuning the noise model using the proposed control relevant method is presented in this chapter[1].

3.1 Introduction

In the last few decades a number of researchers have argued that high closed loop performance needs identification methods that are commensurate with the control

[1]A part of this chapter was presented at AIChE 2001, Halifax (Patwardhan *et al.* 1999*b*). A condensed version of this chapter will appear in Journal of Process Control (Gopaluni *et al.* 2003*a*).

objective (Gevers 2002, Jørgensen and Lee 2001, Veres and Wall 1999). Depending on the control algorithm and the closed loop performance objective function, there are different ways of bringing in synergy between identification and control algorithms (Gevers 1993, Veres and Wall 1999). In this chapter, it is argued that for model predictive controllers this synergy can be realized by minimizing multi-step ahead prediction errors in the identification step instead of minimizing one-step ahead prediction errors as is normally done (Greco *et al.* 1984, Shook *et al.* 1992, Huang and Wang 1999, Rossiter and Kouvaritakis 2001).

Model predictive controllers minimize the control errors over a prediction horizon based on the predicted outputs. Therefore, a significant mismatch between the predicted and true outputs over the prediction horizon can distort the optimization problem in such a way that the resultant solution is sub-optimal. The predictor used in model predictive controllers is a function of the model. Hence, it is important to choose the model carefully so that the resultant closed loop performance is good. We show that a better designed performance can be attained by obtaining a model from minimization of multi-step ahead prediction errors in the identification stage.

Under the assumption that the true process consists of a deterministic part driven by the manipulated variables and a stochastic part driven by white noise, the quality of the predictor depends on the quality of the deterministic and the stochastic parts of the process model. The optimal one-step or multi-step ahead predictor for a given process is the predictor obtained by using the true deterministic and stochastic structures of the process. However, it is generally difficult to know the structure of the true process model and obtain the optimal predictor. The best we can do is to search for the "optimal" predictor over the set of models that we are interested in. As a result, identification is essentially an exercise in approximating the true process model in some "optimal" manner. The question then is : what is "optimal"? It depends on the intended end use of the predictor. Since the controller studied in this chapter is a model predictive controller, we use an optimal multi-step ahead predictor.

There has been a fair amount of work done to this end. The MUSMAR (multi-step Multivariable Adaptive Regulator) algorithm by Greco *et al.* (1984) was one of the earliest works reported in this area. The authors' of this paper are concerned with obtaining better convergence properties and modelling errors by using separate models for each prediction horizon in the context of adaptive control. In order to improve the quality of the predictions over large prediction horizons, they identify different models for each prediction horizon while performing adaptive control. An identification algorithm for minimizing multi-step ahead prediction errors by using separate models for each

prediction horizon was recently proposed by Rossiter and Kouvaritakis (2001). They develop a least squares algorithm to solve the identification problem for ARX type models. They also provide SISO and MIMO examples to show that the models obtained by using their method have better prediction properties. One of the disadvantages of this method is that it can result in identification of a large number of model parameters especially for multi input - multi output processes with large prediction horizons, thus leading to poor statistical properties of the estimated parameters. Moreover, they do not identify a noise model which is crucial for good multi-step ahead predictions.

Huang and Wang (1999) discuss the MPC-relevant identification method in the context of data pre-filtering. They show that the multi-step prediction error minimization algorithm can be reduced to that of one-step ahead prediction error method by filtering the inputs and outputs with a filter that depends on the noise model. Expressions for closed loop multi-step ahead predictors are also derived in their chapter. They also provide simulation examples to show that a predictive controller under no constraints would perform better with a model developed for multi-step ahead predictions.

As mentioned earlier, the accuracy of the predictions depends both on the deterministic and the stochastic part (noise model) of the estimated model. This chapter deals mainly with the finite impulse response models (FIR) and thereby making the bias error in the deterministic part negligible. Once a reasonably good deterministic part is estimated, the quality of the predictor depends on the quality of the noise model. Hence, the performance of the controller under closed loop is directly related to the noise model. We present a method to tune the noise model for optimal multi-step ahead predictions under both open and closed loop conditions.

The main contributions of this chapter are :

- to show that the designed performance with the model obtained from the proposed method is at least as good as any other model
- to show that optimal multi-step ahead predictors provide better predictions over the prediction horizon compared with the optimal one-step ahead predictors whenever the model is a reduced order approximation of the actual process
- reformulation of MPC relevant identification (MRI) strategy for FIR models to eliminate bias errors in the deterministic portion of the process model. Futhermore, it is shown that the accuracy of multi-step ahead predictions depend on the quality of the noise model;
- a study of properties of this identifier in the frequency domain under both open and

closed loop conditions;

- experimental evaluation of the results discussed in this chapter.

Before we proceed with the main results of this chapter, we would like to emphasize that the results presented in this chapter can be extended to multivariable systems. The extensions are technically straightforward. For the sake of simplicity, we have avoided introducing cumbersome notation involved in dealing with multivariable systems. The work presented here is the first step in the direction of this new area of research. There are many more interesting and potentially important issues that need attention when dealing with multivariable systems. However, we have provided a MIMO example in this chapter to make this point.

3.2 Preliminaries

Model predictive controllers (MPC) are widely used in the process industry due to their simplicity and ability to handle constrained multivariable processes. Unlike minimum variance controllers or generalized minimum-variance controllers, model predictive controllers minimize an objective function that is nonmyopic. In other words, the controller minimizes the sum of P-step ahead control errors instead of one-step ahead control errors as is normally done. On the other hand many commonly used identification algorithms do not take this fact into account. A new overall identification criterion, which takes into account the nonmyopic nature of the controller, was defined in Shook *et al.* (1991) and further explored in the context of Generalized Predictive Control (GPC) in Shook *et al.* (1992). This new objective function is control relevant in the sense that it minimizes the sum of P-step ahead prediction errors. The same method has been extended to prediction error models (PEM) in Huang and Wang (1999). In the following paragraphs, a justification is provided for the new approach of minimizing multi-step ahead predictions.

The following standard notation is adopted throughout this work. Given a stochastic function $f(t)$, we define

$$\bar{E}f(t) \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ef(t) \tag{3.1}$$

where E is the expectation operator with the expectation taken only with respect to the noise. The function, $f(t)$, can depend on random variables other than the noise, such as random input. In the rest of this chapter, it is assumed that the expectation operator

is defined with respect to the $\sigma - algebra$[2] associated with the noise. It is also assumed that the above limit exists wherever the operator $\bar{E}$ is used. This limit must exist in our case since we are going to deal with quasi-stationary signals (Ljung 1999).

Considering that the closed loop performance is measured by the squared sum of control errors, the achieved performance is defined by (N is the number of data points)

$$\begin{aligned} J_{ach} &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(r(t) - y(t))^2 \\ &= \bar{E}(r(t) - y(t))^2 \end{aligned} \tag{3.2}$$

It is assumed that the above limit exists. Neglecting the tail/end effects of the control errors (it is true since in most cases $P << N$), we have

$$\begin{aligned} J_{ach} &\approx \frac{1}{P}\bar{E}\left[\sum_{j=1}^{P}(r(t+j) - y(t+j))^2\right] \\ &= \lim_{(N-P)\to\infty} \frac{1}{(N-P)P} \sum_{t=1}^{N-P}\sum_{j=1}^{P} E(r(t+j) - y(t+j))^2 \end{aligned} \tag{3.3}$$

A proof of equation (3.3) is provided in appendix 3.A . Therefore, minimizing (3.2) is equivalent to minimizing (3.3). Now expanding (3.3) and ignoring the factor $1/P$ for the sake of simplicity, we get

$$\begin{aligned} J_{ach} &\approx \bar{E}\sum_{j=1}^{P}\left(r(t+j) - \hat{y}(t+j|t) - \varepsilon_j^p(t)\right)^2 \\ &= \bar{E}\sum_{j=1}^{P}\left(r(t+j) - \hat{y}(t+j|t)\right)^2 + \bar{E}\sum_{j=1}^{P}\left(\varepsilon_j^p(t)\right)^2 \\ &\quad -2\bar{E}\sum_{j=1}^{P}\left(r(t+j) - \hat{y}(t+j|t)\right)\left(\varepsilon_j^p(t)\right) \end{aligned} \tag{3.4}$$

where $\hat{y}(t+j|t)$ is the optimal j-step ahead predictor at time instant, t. Any certainty equivalence formulation of model predictive controller minimizes only the first term in the above equation and as a result it is sub-optimal from a global perspective. Therefore any identification algorithm which minimizes the remaining terms can be expected to be "optimal" in some sense. This idea forms the crux of this chapter. In Huang and Wang (1999) and Rossiter and Kouvaritakis (2001) the second term is minimized to obtain a

[2] $\sigma - algebra$, in nontechnical terms, can roughly be defined as the set of information corresponding to a particular stochastic variable.

MPC relevant model. The third term which represents the cross correlation between the identification errors and the control errors can be handled by performing iterative model tuning (Gevers 2002). In the rest of the chapter, the following objective function which minimizes multi-step ahead prediction errors is referred to as MPC relevant identifier (MRI)

$$\begin{aligned} J^{(P)} &= \frac{1}{P}\bar{E}\sum_{j=1}^{P}\left(\varepsilon_j^p(t)\right)^2 \\ &= \frac{1}{P}\bar{E}\sum_{j=1}^{P}\left(y(t+j)-\hat{y}(t+j|t)\right)\right)^2 \end{aligned} \tag{3.5}$$

The idea behind using a multi-step ahead prediction error objective function for identification is to obtain a model that results in better multi-step ahead predictions on an average. But then it is important to show that the performance improves with the model obtained using the MRI objective function. The following result shows that the designed performance at each sample instant is on an average better with the MRI model.

Theorem 3.2.1 *Consider two models, $\mathcal{S}_1$ and $\mathcal{S}_2$. Let us suppose that $\mathcal{S}_1$ is obtained using the MRI objective function and that $\mathcal{S}_2$ is obtained using the traditional one-step ahead prediction error method. Define the achieved and designed performances with a model, $\mathcal{S}$ at sample instant, t, as follows :*

$$J_{ach}^{(P)}(t,u_f(t)) := E\left[\sum_{j=1}^{P}(r(t+j)-y(t+j))^2\middle|\mathcal{F}_t\right] \tag{3.6}$$

$$J_{des}^{(P)}(t,u_f(t),\mathcal{S}) := E\left[\sum_{j=1}^{P}(r(t+j)-\hat{y}(t+j|t))^2\middle|\mathcal{F}_t\right] \tag{3.7}$$

where

$$\mathcal{F}_t := \sigma - algebra\{u(-\infty),y(-\infty),\cdots,u(t),y(t),u(t+1),u(t+2),\cdots,u(t+P)\}$$

and $u_f(t)$ is the vector of future inputs and $y(t)$ are the input and output values at time instant, t. Let $\hat{y}_1(t)$ denote the predictor obtained using $\mathcal{S}_1$ and let $\hat{y}_2(t)$ be similarly defined. Also, assume that the set point $r(t)$ is constant over the prediction horizon, i.e.,

$$r(t+j) := r \qquad for\ 1 \le j \le P \tag{3.8}$$

where r is some constant and that the prediction errors with both the models have mean zero i.e.,

$$\begin{aligned} E[\hat{y}_1(t+j|t) - y(t+j)|\mathcal{F}_t] &= 0 \quad \forall\ t, j \\ E[\hat{y}_2(t+j|t) - y(t+j)|\mathcal{F}_t] &= 0 \quad \forall\ t, j. \end{aligned}$$

Then the following is true

$$|J_{ach}^{(P)}(t, u_f(t)) - J_{des}^{(P)}(t, u_f(t), \mathcal{S}_1)| \leq |J_{ach}^{(P)}(t, u_f(t)) - J_{des}^{(P)}(t, u_f(t), \mathcal{S}_2)| \quad (3.9)$$

Proof: Proof is provided in appendix 3.B .

□

Remark 3.2.1 The above theorem shows that the designed performance at each sample instant for a given input sequence is closer to the true performance when an MRI model is used. This forms the motivation for exploring the new objective function defined in this chapter for identification. In the paper by Gevers (1993) also, an identification objective function that minimizes the difference between achieved and designed performances is used. However, one would like to have better achieved performance. This, in our opinion, needs an iterative approach which is discussed in a later chapter.

The above result is valid only when the input sequences used in evaluating $J_{des}^{(P)}(t, \mathcal{S}_1)$ and $J_{des}^{(P)}(t, \mathcal{S}_2)$ are the same. However, when a model predictive controller is implemented, the optimal input sequence generated at each sample instant depends on the model. The following result shows that, the optimum achieved with the MRI model is better than or equal to that achieved using one-step ahead prediction error model.

Theorem 3.2.2 *Suppose that*

$$\begin{aligned} u_f^{(1)}(t) &:= \min_{u_f(t)} J_{des}^{(P)}(t, u_f(t), \mathcal{S}_1) \\ u_f^{(2)}(t) &:= \min_{u_f(t)} J_{des}^{(P)}(t, u_f(t), \mathcal{S}_2) \end{aligned} \quad (3.10)$$

Then under the assumptions and the definitions provided in Theorem 3.2.1, the following is true

$$|J_{ach}^{(P)}(t, u_f^{(1)}(t)) - J_{des}^{(P)}(t, u_f^{(1)}(t), \mathcal{S}_1)| \leq |J_{ach}^{(P)}(t, u_f^{(2)}(t)) - J_{des}^{(P)}(t, u_f^{(2)}(t), \mathcal{S}_2)| \quad (3.11)$$

where $u(1)_f$ and $u(2)_f$ are the corresponding vectors of the future inputs.

Proof: The proof follows directly from Theorem 3.2.1.

□

Corollary 3.2.1 *The designed performance with an MRI model at each sample instant is, on an average, better than or equal to that obtained using one-step ahead prediction error model.*

Proof: Follows from Theorem 3.2.1 and Theorem 3.2.2

□

It is useful to note that the control objective function

$$\bar{E}\sum_{j=1}^{P}\left(r(t+j)-\hat{y}(t+j|t)\right)^2 \tag{3.12}$$

can be modified to account for any time delay in the process. If T_d is the time delay, then generally a modified objective function of the form

$$\bar{E}\sum_{j=T_d}^{P}\left(r(t+j)-\hat{y}(t+j|t)\right)^2 \tag{3.13}$$

is used in the controller in which case the multi-step ahead identification objective function can also be modified to

$$\bar{E}\sum_{j=T_d}^{P}\left(\varepsilon_j^p(t)\right)^2. \tag{3.14}$$

However, this method assumes that the time delay is known *a priori*. It is also possible to add the input weighting to the closed loop performance objective function. Input weighting term will not affect the identification procedure since it is independent of the identification parameters.

In the next three sections a number of properties of the MPC relevant identifier (sometimes called control relevant identifier) are presented.

3.3 MPC relevant identification

Invariably, all identified models have certain bias and variance errors associated with them. Most of the currently available control relevant identification methods try to shape these errors in such a way that high closed loop performance can be achieved.

The bias errors in the deterministic part of the model can be minimized by using FIR models (in most cases) and this leaves us with only variance errors. Moreover, the widely used model predictive controllers use step response coefficients in the control algorithm. Therefore it is instructive to study the properties of a MPC relevant identifier based on FIR models. A number of interesting properties of this method of identification in relation to tuning the noise model are presented in this section and in the following sections. But before that the importance of the noise model in MRI is discussed in the following section.

3.3.1 Importance of the noise model in MPC Relevant Identification

A good noise model is vital for good multi-step ahead predictions. In this section it is shown that in the absence of a noise model, there is negligible difference between the MPC relevant method and the one-step ahead prediction error methods. Consider the following FIR representation of a process.

$$y(t) = h^T u_t + e(t) \tag{3.15}$$

where $u_t = [u(t-1)\cdots u(t-n)]$, n is the number of impulse response coefficients, N the data length and P is the prediction horizon. $e(t)$ is a white noise sequence with zero mean and a certain variance. h is the vector of true impulse response coefficients. Since the measurements are corrupted with pure white noise, the noise model is unity. Let us denote the finite data equivalent of (3.5) by $J_N^{(P)}$. Hence, the MRI objective function for this model is given by

$$\begin{aligned} J_N^{(P)}(\hat{h}) &= \frac{1}{P(N-P)} \sum_{t=0}^{N-P} \sum_{j=1}^{P} [y(t+j) - \hat{y}(t+j|t)]^2 \\ &= \frac{1}{P(N-P)} \sum_{t=0}^{N-P} \sum_{j=1}^{P} \left[y(t+j) - \hat{h}^T u_{t+j}\right]^2 \end{aligned}$$

where $\hat{h}$ is the vector of estimated impulse response coefficients. Since noise model is unity, the following is true

$$\hat{y}(t+j/t) = \hat{y}(t+j/t+1) = \cdots = \hat{y}(t+j/t+j-1) = \hat{h}^T u_{t+j} \tag{3.16}$$

i.e., the predictions into the future do not depend on the past prediction errors or on the past measured outputs. The above equation follows from the fact that $e(t)$ is

white with zero mean. Therefore, expanding the above double summation and denoting $(y(t) - \hat{h}^T u_t)$ by $\hat{e}(t)$, the objective function becomes

$$J_N^{(P)}(\hat{h}) = \frac{1}{P(N-P)}\left[\hat{e}(1)^2 + 2\hat{e}(2)^2 + \cdots + P\sum_{t=P}^{N-P+1}\hat{e}(P)^2 + (P-1)\hat{e}(N-P+2)^2 + \cdots + 2\hat{e}(N-1)^2 + \hat{e}(N)^2\right] \tag{3.17}$$

Comparing it with the one-step ahead prediction error objective function

$$J_N^{(1)} = \frac{1}{N}\left[\hat{e}(1)^2 + \cdots + \hat{e}(N)^2\right]$$

it can be seen that as long as $N >> P$ i.e., the data length is much larger than the prediction horizon,

$$J_N^{(P)}(\hat{h}) \approx J_N^{(1)}(\hat{h}) \quad \forall \ \hat{h}. \tag{3.18}$$

This implies that there is little difference between the minimization of one-step ahead and P-step ahead errors in the absence of a noise model. Therefore a noise model is crucial for the MPC relevant methods especially when using FIR models. Since any realistic process model has a noise model associated with it, MRI methods have an advantage over the ordinary methods in identifying better multi-step ahead predictors.

The difference between the one-step ahead prediction method and the MRI method is a weighting matrix. This weighting matrix can also be interpreted as a filter on the one-step ahead predictions. The MPC relevant identification objective function can be rewritten as

$$J_N^{(P)}(\hat{h}) = [\hat{e}(1)\hat{e}(2)\cdots\hat{e}(N)]\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{e}(1) \\ \hat{e}(2) \\ \vdots \\ \hat{e}(N) \end{bmatrix}.$$

representing the weighting matrix at the centre by Γ_N and the error vector by β, we get

$$J_N^{(P)}(\hat{h}) = (\Gamma_N^{1/2}\beta)^T(\Gamma_N^{1/2}\beta)$$

and the filtered error

$$\beta_f = (\Gamma_N^{1/2}\beta).$$

In this particular illustration an FIR model was used. For any general model, it is possible to show that the MRI problem is a weighted least squares problem with the weights dependent on some nonlinear function of the noise model (see appendix 3.C).

In summary, if there is no noise model, MPC relevant methods are similar to the one-step ahead prediction error methods for FIR models. Since FIR models have negligible bias errors, MRI methods try to shape the variance errors. If the process is corrupted with pure white noise it is not possible to shape these variance errors. These methods have a distinct advantage over traditional identification methods if the effect of noise model on the output is significant *i.e.*, if the signal to noise ratio is small.

3.3.2 Frequency domain analysis of MRI

The MRI can be treated as a method to enhance the accuracy of the model in the frequency ranges that are important for the minimization of multi-step ahead prediction errors. A frequency domain interpretation of this method is useful in gaining better insight into the trade-off between bias and variance errors. An analysis of this method under both open and closed loops is presented in this section. It is shown that the MRI method is equivalent to the traditional identification methods but with the data pre-filtered using a filter that depends on the noise model. It is also shown that the MRI methods have an advantage over traditional methods only when there is model plant mismatch (mismatch can be either in the plant model or the noise model). A number of other properties of this method are derived in this section.

Open loop Identification

Consider the following process with the true model given by

$$\mathcal{S} \;:\; y(t) = G(q)u(t) + H(q)e(t)$$

and an identified model $\hat{G}(q,\theta)$ and $\hat{H}(q,\theta)$ where θ is the parameter vector [3]. Then the j-step ahead optimal predictor is given by (Ljung 1999)

$$\hat{y}(t+j/t) \;=\; \hat{W}_j\hat{G}_p u(t+j) + (1-\hat{W}_j)y(t+j)$$

where

$$\begin{aligned}\hat{W}_j &= \hat{F}_j\hat{H}^{-1} \\ \hat{F}_j &= \sum_{i=0}^{j-1}\hat{g}_i q^{-i} \;; \qquad \hat{F}_1 = \hat{g}_0 = 1\end{aligned}$$

[3]Note that $\hat{G}$ can be an impulse response model

and $\hat{g}(i)$ are the impulse response coefficients of $\hat{H}$. It is assumed that all the inputs and the outputs are quasi-stationary signals. We also assume that $\hat{G}$ and $\hat{H}$ are independent of the data length by defining the identified models as

$$\hat{G} = \lim_{N\to\infty} \hat{G}^N \tag{3.19}$$

$$\hat{H} = \lim_{N\to\infty} \hat{H}^N \tag{3.20}$$

where $\hat{G}^N$ and $\hat{H}^N$ are models identified from a data of length N. In the rest of this chapter all transfer functions with a superscript 'N' denote the identified models from a data set of length 'N'. Transfer function estimates without the superscript are the model estimates as $N \to \infty$. Then we have the following result which separates the bias and variance errors of the estimated model.

Proposition 3.3.1 *The MPC relevant identifier with multi-step ahead prediction errors of the form*

$$J^{(P)} = \frac{1}{P}\bar{E}\sum_{j=1}^{P}(y(t+j) - \hat{y}(t+j/t))^2$$

has a conditional expected value, under open loop conditions (i.e., u_t is independent of noise), given by

$$J^{(P)} = \frac{1}{P}\left(\sum_{j=1}^{P}\bar{\mu}_j^2 + \sum_{j=1}^{P}\bar{\sigma}_j^2\right) \tag{3.21}$$

where $\bar{\mu}_j$ is a function of both process and noise models and σ_j is a function of the noise model only.

Proof: Consider the j-step ahead prediction error

$$\begin{aligned}(y(t+j) - \hat{y}(t+j/t)) &= G_p u(t+j) + G_L e(t+j) - \hat{W}_j^N \hat{G}^N u(t+j) \\ &\quad -(1-\hat{W}_j^N)(G_p u(t+j) + G_L e(t+j)) \\ &\triangleq \varepsilon_{t,j}^p\end{aligned}$$

Now taking the expected value of the prediction errors with respect to the noise, e_t (*i.e.*, assuming that the inputs are known and independent of noise) we get

$$\begin{aligned}E\left[\varepsilon_{t,j}^p | u\right] &= \hat{W}_j^N(G - \hat{G}^N)u_{t+j} \triangleq \hat{W}_j^N \tilde{G}^N u_{t+j} \\ &\triangleq \mu_j(t)\end{aligned}$$

Therefore, given the process model and the input excitation, $\varepsilon_{t,j}^p$ is normally distributed with mean μ_j and variance σ_j^2 where

$$\begin{aligned}\sigma_j^2 &= Var(G_L e_{t+j} - (1 - \hat{W}_j^N) G_L e_{t+j}) \\ &= Var(\hat{W}_j^N G_L e_{t+j}) \\ &= \|\hat{W}_j^N H\|_2^2 \sigma_e^2 = \|\hat{F}_j^N \frac{H}{\hat{H}^N}\|_2^2 \sigma_e^2 \end{aligned} \tag{3.22}$$

where $g(i)$ are the impulse response coefficients of the true noise model H and σ_e^2 is the noise variance. Since $u(t)$ is quasi-stationary, the following limits exist

$$\begin{aligned}\bar{\mu}_j^2 &\triangleq \bar{E}\mu_j^2 \\ \bar{\sigma}_j^2 &\triangleq \bar{E}\sigma_j^2\end{aligned}$$

Therefore the MPC relevant identification objective function is[4]

$$J^{(P)} = \frac{1}{P}\left(\sum_{j=1}^{P} \bar{\mu}_j^2 + \sum_{j=1}^{P} \bar{\sigma}_j^2\right) \tag{3.23}$$

□

Corollary 3.3.1 *If $\hat{H} = H$ then the variance of P-step prediction errors is*

$$\frac{1}{P}\sum_{j=1}^{P} \bar{\sigma}_j^2 = \frac{1}{P}\left[P + (P-1)g_1^2 + (P-2)g_2^2 + \cdots + g_{P-1}^2\right]\sigma_e^2 \tag{3.24}$$

Proof: Follows directly from (3.22).

□

Corollary 3.3.2 *The minimum P-step prediction error variance is $P\sigma_e^2$ but it is not realizable with any monic noise filter.*

Remark 3.3.1 If we are dealing with FIR models and if the bias errors are negligible *i.e.*, $\tilde{G} \approx 0$ then the expected value of the MRI objective function reduces to

$$J^{(P)} = \frac{1}{P}\sum_{j=1}^{P} \bar{\sigma}_j^2 \tag{3.25}$$

In such a case the P-step ahead prediction errors depend only on the variance term which depends only on the noise model chosen and the true noise model. This suggests

[4]From the fact that if X is normally distributed with mean μ and variance σ^2 then $E[X^2] = \mu^2 + \sigma^2$

that it is important to tune the noise model in an "optimal" way. The bias, if present, in the identified model, is reflected by the first term in (3.21). It is worth noting that the minimum P-step ahead prediction error occurs if

$$\hat{W}_j = H \quad \forall\, j \in \{1, 2, \cdots, \} \tag{3.26}$$

but since Eqns.3.26 can not be true simultaneously, we can never achieve this minimum with a single noise model and a causal predictor. The same can possibly be achieved by using different noise models for each j (at the cost of many fold increase in the model parameters). Using (3.21), it can be shown that

$$J^{(P)} = \frac{1}{P}\sum_{j=1}^{P}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[|\hat{W}_j\tilde{G}|^2\Phi_u(\omega) + |\hat{W}_jH|^2\Phi_e(\omega)\right]d\omega \tag{3.27}$$

where Φ_u and Φ_e are the input and noise spectra respectively. The argument, $e^{j\omega}$, of the transfer functions is not explicitly shown in the above equations. The first term in (3.27) represents the bias error and the second term represents the variance errors.

Similarly, for $P = 1$, we have the one-step ahead prediction error objective function

$$J^{(1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[|\hat{H}^{-1}\tilde{G}|^2\Phi_u(\omega) + |\hat{H}^{-1}H|^2\Phi_e(\omega)\right]d\omega \tag{3.28}$$

Therefore MRI methods are different from the other identification methods in the sense that they use a data pre-filter of the form $\sum_{j=1}^{P}|\hat{F}_j(\omega)|^2$ (compare (3.27) with (3.28)). From another perspective, the MRI is equivalent to changing the input spectrum to $\sum_{j=1}^{P}|\hat{F}_j(\omega)|^2\Phi_u(\omega)$ and the noise spectrum to $\sum_{j=1}^{P}|\hat{F}_j(\omega)|^2\Phi_e(\omega)$. Hence, asymptotic variance of the estimated process models under negligible bias errors is given by (Ljung 1999)

$$Cov([\hat{G}\ \ \hat{H}]^T) \sim \frac{n}{N}\frac{\sum_{j=1}^{P}|\hat{F}_j(\omega)|^2|\hat{H}|^2\Phi_e(\omega)}{\sum_{j=1}^{P}|\hat{F}_j(\omega)|^2\Phi_u(\omega)} = \frac{n}{N}\frac{|\hat{H}|^2\Phi_e(\omega)}{\Phi_u(\omega)} \tag{3.29}$$

where 'n' is the number of unknown parameters. The MRI method and the one-step ahead prediction error methods result in the same asymptotic variance if the bias errors are negligible.

By minimizing P-step ahead prediction errors, a model that can provide the best predictions over a particular prediction horizon is obtained. A model obtained by minimizing one-step ahead prediction errors may not be optimal for multi-step ahead predictions. It is clear that we need to identify a good noise model to design the data pre-filter, which in many cases is not an easy task. Hence, the "performance" of the predictor depends on the quality of the noise model.

It is of interest to know if the proposed method always provides a model that is different from an optimal model obtained by minimizing one-step ahead prediction errors. This is not true in general. For certain structures of the noise model, the optimal models obtained may not differ much in both the cases. The following proposition explains this fact.

Proposition 3.3.2 *one-step ahead prediction algorithms for identification and the MPC relevant identification algorithms would lead to the same model if the noise model has the following structure*

$$\hat{H} = 1 + \sum_{i=P}^{\infty} \hat{g}_i e_{t-i} \tag{3.30}$$

Proof: (3.27) and (3.28) are one and the same if

$$\hat{H} = 1 + \sum_{i=P}^{\infty} \hat{g}_i e_{t-i} \tag{3.31}$$

i.e., if

$$\hat{F}_j = 0 \quad \forall j \in \{2, \cdots, P\} \tag{3.32}$$

□

Remark 3.3.2 In the unlikely case of the estimated noise structure being same as the one in (3.30), the MRI methods result in the same model as the ordinary one-step ahead prediction error methods. (3.30) can then be written as

$$\hat{H} = 1 + q^{-P}\hat{H}' \tag{3.33}$$

where $\hat{H}'$ is a monic filter. This type of a noise model has a selective memory. At any instant, t, it remembers the immediate value of the input and the inputs before time $t - P + 1$.

In one of the introductory sections we have claimed that the MRI method and the traditional identification methods result in the same maximum likelihood estimate of the model if the true model structures are known. This claim is proved in the following proposition.

Proposition 3.3.3 : *The multi-step ahead prediction error objective function has a minimum at $\hat{G} = G$ and $\hat{H} = H$ and the minimum is given by*

$$\frac{1}{P}\sum_{j=1}^{P} \bar{\sigma}_j^2 = \frac{1}{P}\left[P + (P-1)g_1^2 + (P-2)g_2^2 + \cdots + g_{P-1}^2\right]\sigma_e^2 \tag{3.34}$$

Proof: Let us define the following

$$\begin{aligned} H &\triangleq F_j + H_j \\ \hat{H} &\triangleq \hat{F}_j + \hat{H}_j \end{aligned} \tag{3.35}$$

then

$$\begin{aligned} \varepsilon^p_{t,j} &= \hat{W}_j \tilde{G}_p u(t+j) + \hat{W}_j H e(t+j) \\ &= \hat{W}_j \tilde{G}_p u(t+j) + \left[\frac{H_j}{H} - \frac{\hat{H}_j}{\hat{H}}\right] H e(t+j) + F_j e(t+j) \end{aligned} \tag{3.36}$$

Noting that the second term depends only on the past noise and that $F_j e_{t+j}$ depends on future noise, we get

$$\begin{aligned} \underset{\theta}{argmin}\ J^{(P)} &= \underset{\theta}{argmin} \left\{ \frac{1}{P}\sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} \left[|\hat{W}_j \tilde{G}|^2 \Phi_u(\omega) + \left| \frac{H_j}{H} - \frac{\hat{H}_j}{\hat{H}} \right|^2 |H|^2 \Phi_e(\omega) \right] d\omega \right. \\ &\quad \left. + \frac{1}{P}\sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} |F_j|^2 \Phi_e(\omega) d\omega \right\} \\ &= \frac{1}{P}\left[P + (P-1)g_1^2 + (P-2)g_2^2 + \cdots + g_{P-1}^2\right] \sigma_e^2 \end{aligned} \tag{3.37}$$

θ being the parameter vector. The minimum then occurs at $\hat{G} = G$ and $\hat{H} = H$

□

Remark 3.3.3 The above result shows that if we know the structure of the true process model and the true noise model then the optimal multi-step or one-step ahead predictions are given by the true models. But in any real process, the true structures are seldom known and therefore it is important to identify a multi-step ahead prediction model.

The optimal solution of the identification problem is one which minimizes the objective function in (3.27). Since the process and noise models occur together in these expressions, the minimization problem is highly nonlinear. In many industrial model predictive controllers the noise model is fixed *a priori* (*e.g.*, an integrator). In such a case the estimated noise model is fixed and the best process model possible for that particular noise model needs to be estimated. In the next two sections an analysis of this case and the case where the noise model is not fixed *a priori* are presented.

Fixed noise model

When the noise model is fixed (*i.e.*, optimize the objective function only with respect to $\hat{G}$), the MRI problem reduces to that of minimizing a modified form of the objective function $J^{(P)}$

$$\begin{aligned} J_{mod}^{(P)} &= \frac{1}{P}\sum_{j=1}^{P}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[|\hat{W}_j\tilde{G}|^2\Phi_u(\omega)\right]d\omega \\ &= \frac{1}{2\pi P}\int_{-\pi}^{\pi}\left[|\hat{H}^{-1}\tilde{G}|^2\Phi_u(\omega)\right]d\omega + \frac{1}{2\pi P}\int_{-\pi}^{\pi}\left[|1+\hat{g}_1e^{-i\omega}|^2|\hat{H}^{-1}\tilde{G}|^2\Phi_u(\omega)\right]d\omega \\ &\quad + \cdots\cdots + \frac{1}{2\pi P}\int_{-\pi}^{\pi}\left[|1+\sum_{k=1}^{P-1}\hat{g}_ke^{-ik\omega}|^2|\hat{H}^{-1}\tilde{G}|^2\Phi_u(\omega)\right]d\omega. \end{aligned} \tag{3.38}$$

Traditional identification methods minimize only the first term in the above equation but on the other hand (3.21) shows that it is important to minimize other terms as well to obtain minimum possible variance in the prediction errors. If G lies in the set $\mathcal{G}$ of models over which the optimization is done, then the estimated model is going to be same as the true model and therefore there are no incentives in using MRI models. Nevertheless, it is not always practical to assume that $G \in \mathcal{G}$ since in certain cases reduced complexity models are of interest and that is when MPC relevant identification method helps us in distributing the bias. The issues involved in bias distribution of process models identified using the proposed method are not discussed in this chapter (readers are referred to (Gopaluni *et al.* 2002*a*)). Since this article deals exclusively with FIR models, $G \in \mathcal{G}$ and therefore, the noise model plays a vital role in determining the performance of the predictor (and therefore the controller).

Unknown noise model

If the noise model is not known, then it is necessary to identify $\hat{G}$ and $\hat{H}$ simultaneously. If $G \in \mathcal{G}$ then the problem can be solved in two steps (see section 4 for details). In the first step, an unbiased model, $\hat{G}$ can be obtained by fixing the noise model arbitrarily (c.f (3.27)). Once an unbiased process model is obtained, the first term in (3.27) disappears and therefore an MPC relevant noise model can be obtained by minimizing the following objective function

$$J^{(P)} = \sum_{j=1}^{P}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[|\hat{W}_jH|^2\Phi_e(\omega)\right]d\omega. \tag{3.39}$$

From (3.37) the optimal noise model is the true noise model. If the true structure of the noise model is known then, clearly, the optimal estimate of the noise model is H

itself. But in any practical situation, it is difficult to know the true structure of the noise model. As a result, from (3.39), the best noise model among the set of models over which the optimization is done, will be a function of the true noise model.

3.3.3 Closed loop Identification

There is extensive literature on closed loop identification (Forssell and Ljung 1999, MacGregor and Fogal 1995, Huang and Shah 1997) based on minimizing one-step ahead prediction errors. In this chapter closed loop identification based on minimizing multi-step ahead prediction errors is of concern. Closed loop identification methods can broadly be classified into: direct closed loop identification, indirect closed loop identification and joint input-output identification. The bias distribution of the identified model depends on the method of closed loop identification. In this chapter, we have only considered the direct closed loop identification method. In the direct closed loop identification method, inputs and outputs are used to identify the process model without considering the cross-correlation between inputs and outputs. This method generally works well under high signal to noise ratios. The expressions for the objective function have to be modified under closed loop to account for the correlation between the input and the noise.

The MRI objective function under closed loop can be shown to be

$$\begin{aligned} J^{(P)} &= \frac{1}{2\pi}\sum_{j=1}^{P}\int_{-\pi}^{\pi}\left|\frac{\hat{W}_j\tilde{G}C}{1+CG}\right|^2\Phi_r(\omega)+\left|\hat{W}_jH-\frac{\hat{W}_j\tilde{G}_PCH}{1+CG}\right|^2\Phi_e(\omega)d\omega \\ &= \frac{1}{2\pi}\sum_{j=1}^{P}\int_{-\pi}^{\pi}\left|\hat{W}_j\tilde{G}\right|^2|CS|^2\Phi_r(\omega)+\left|\hat{W}_jH\right|^2\frac{|S|^2}{|\hat{S}|^2}\Phi_e(\omega)d\omega \qquad (3.40) \\ &= \frac{1}{2\pi}\sum_{j=1}^{P}\int_{-\pi}^{\pi}\left|\hat{W}_j\tilde{G}\right|^2|CS|^2\Phi_r(\omega)+\left|\frac{H_j}{H}-\frac{\hat{H}_j}{\hat{H}}\right|^2\left|\frac{S}{\hat{S}}H\right|^2+\left|F_j\frac{S}{\hat{S}}\right|^2\Phi_e(\omega)d\omega \qquad (3.41) \end{aligned}$$

where $\hat{S}$ and S are the true and estimated sensitivity functions defined by

$$S \triangleq \frac{1}{1+CG} \qquad \hat{S} \triangleq \frac{1}{1+C\hat{G}}. \qquad (3.42)$$

and Φ_r is the set point spectrum.

As in the open loop case an unbiased FIR estimate of G can be obtained by fixing the noise model, in this case to $\hat{S}^{-1}$ (*i.e.*, $\hat{W}_j = \hat{S}$),[5] and then performing the optimization.

[5]Strictly speaking $\hat{W}_j \neq \hat{S}$ for all j but this substitution helps in decoupling the noise model from the process model in the identification step.

This particular parametrization makes the second term in (3.40) independent of the process model, $\hat{G}$. Hence, minimization of the identification objective function with the above parametrization results in unbiased model estimate. On the other hand, if $G \notin \mathcal{G}$ the model bias has to be distributed in such a way that the above objective function is minimum. In case of FIR models, once the unbiased estimate of G is estimated, an optimal noise model for good multi-step ahead predictions can be obtained by minimizing the modified objective function (since the first term in (3.40) disappears)

$$J_{mod}^{(P)} = \frac{1}{2\pi} \sum_{j=1}^{P} \int_{-\pi}^{\pi} \left| \hat{W}_j \right|^2 \Phi_e(\omega) d\omega. \tag{3.43}$$

The minimum of this objective function clearly occurs at $\hat{H} = H$. At $\hat{H} = H$

$$J_{mod}^{(P)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^{P} |F_j|^2 \Phi_e(\omega) d\omega \tag{3.44}$$

Therefore as in open loop identification, the problem reduces to that of optimizing the above objective function. In this case there is no difference between closed loop MPC relevant identification and open loop MRI for FIR models with the above mentioned noise models.

Proposition 3.3.4 *If $G \in \mathcal{G}$ and if identification is done as in section 3.3.3 then the MPC relevant noise models obtained under both open loop and closed loop experiments are the same.*

3.4 Two-step MPC Relevant Identification

It is clear from the previous section that identifying an "optimal" noise model is the most important step in MPC relevant identification for FIR models. Since FIR models are estimated, bias error is not a problem. Variance errors, on the other hand, can be influenced by the noise model and this has been demonstrated clearly in the aforementioned paragraphs. Any fixed *a priori* noise model would result in an unbiased FIR estimate for the process model under open loop. Therefore, MPC relevant Identification can be carried out by the following two steps.

Algorithm:

1) Fix an *a priori* noise model (e.g., $\hat{H} = 1$). Identify an FIR model based on the assumed noise model.
2) Using the process model obtained in the previous step, identify a noise model by minimizing the prediction errors with respect to the parameters in noise model.

Let the noise model be

$$\hat{H} = \frac{C}{D} = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + \cdots + c_m q^{-m}}{1 + d_1 q^{-1} + d_2 q^{-2} + \cdots + d_m q^{-n}} \tag{3.45}$$

then the impulse response coefficients of this noise model will be functions of c_i's and d_j's for $1 \leq i \leq m$ and $1 \leq j \leq n$ respectively *i.e.*,

$$\hat{g}_j = \hat{g}_j(c_1, \cdots, c_j, d_1, \cdots, d_j) \quad \text{and} \quad \hat{g}_0 \triangleq c_0 \triangleq 1$$

Let us now define a new variable, $\hat{g}_j^{j+1}$ which is obtained by replacing all the c_j's and d_j's in $\hat{g}_j$ by c_{j+1}'s and d_{j+1}'s *i.e.*,

$$\hat{g}_j^{j+1} \triangleq \hat{g}_j(c_1, \cdots, c_{j+1}, d_1, \cdots, d_{j+1})$$

then

$$\hat{g}_j(c_1, \cdots, c_j, d_1, \cdots, d_j) = \hat{g}_{j-1}^{j} - \hat{g}_{j-1} d_1 \quad \text{for} \quad j = 1 \text{ to } P \tag{3.46}$$

Using these recursive relations for the impulse response coefficients it is possible to pose the noise model identification problem as a nonlinear parameter estimation problem.

$$\begin{aligned} \varepsilon_{t,j}(\theta_j) &\triangleq y(t+j) - \hat{y}(t+j|\theta_j) \\ &= \frac{D(q)}{C(q)} \left[\hat{F}_j y_{t+j} - \hat{F}_j u_{t+j} \right] \end{aligned} \tag{3.47}$$

where θ_j is the vector of unknown parameters. Introducing the auxiliary variables

$$\begin{aligned} \nu_{t,j}(\theta_j) &= \left[\hat{F}_j y_{t+j} - \hat{F}_j u_{t+j} \right] \\ &= y_{t+j} + \hat{g}_1 y_{t+j-1} + \cdots + \hat{g}_{j-1} y_{t+1} - u_{t+j} - \hat{g}_1 u_{t+j-1} \cdots - \hat{g}_{j-1} u_{t+1} \end{aligned} \tag{3.48}$$

we can write

$$\varepsilon_{t,j}(\theta_j) = \nu_{t,j}(\theta_j) + d_1 \nu_{t-1,j}(\theta_j) + \cdots + d_n \nu_{t-n,j}(\theta_j) - c_1 \varepsilon_{t-1,j}(\theta_j) + \cdots + c_m \varepsilon_{t-m,j}(\theta_j) \tag{3.49}$$

Substituting (3.48) into (3.49), we find that

$$\varepsilon_{t,j}(\theta_j) = y_{t+j} - \varphi_{t,j}^T \theta_j \tag{3.50}$$

where

$$\begin{aligned}\varphi_{t,j}(\theta_j) &= [-y_{t+j-1}+u_{t+j-1}, -y_{t+j-2}+u_{t+j-2}, \cdots, -y_{t+1}+u_{t+1}, u_{t+j} \\ &\quad \varepsilon_{t-1,j}(\theta_j), \cdots, \varepsilon_{t-m,j}(\theta_j), -\nu_{t-1,j}(\theta_j), \cdots, -\nu_{t-n,j}(\theta_j)]^T \end{aligned} \tag{3.51}$$

and

$$\theta_j = [\hat{g}_1, \cdots, \hat{g}_j, 1, c_1, \cdots, c_m, d_1, \cdots, , d_n]^T \tag{3.52}$$

Note that all the impulse response coefficients of the noise model in θ_j depend on model coefficients according to the relation in (3.46). Defining a new matrix I_j of dimension $j \times P$ we can write

$$\theta_j = I_j \theta \tag{3.53}$$

where

$$\begin{aligned}\theta &= \theta_P \\ &= [\hat{g}_1, \cdots, \hat{g}_P, 1, c_1, \cdots, c_m, d_1, \cdots, , d_n]^T \end{aligned} \tag{3.54}$$

and I_j is a matrix with ones along its principal diagonal and zeros elsewhere. Now the identification problem can be posed as

$$\begin{aligned}&\min_\theta \textstyle\sum_{t=1}^{N-P} \sum_{j=1}^{P} (\varepsilon_{t,j})^2 \\ &\text{s.t} \quad \hat{g}_j(c_1, \cdots, c_j, d_1, \cdots, d_j) = \hat{g}^j_{j-1} - \hat{g}_{j-1} d_j \end{aligned} \tag{3.55}$$

Example 3.4.1 The most common noise model used in generalized predictive controller is of the form $\frac{C(q^{-1})}{1-q^{-1}}$. The impulse response coefficients of this particular structure for noise model have an interesting and simple structure.

$$\begin{aligned}\hat{H} &= 1 + (1+c_1)q^{-1} + (1+c_1+c_2)q^{-2} + \cdots + (1+c_1+\cdots+c_m)q^{-m} \\ &\quad + (1+c_1+\cdots+c_m) \sum_{i=m+1}^{\infty} q^{-i} \\ &= 1 + (1+c_1)q^{-1} + (1+c_1+c_2)q^{-2} + \cdots + (1+c_1+\cdots+c_m)q^{-m} \\ &\quad + (1+c_1+\cdots+c_m) \frac{q^{-(m+1)}}{1-q^{-1}} \end{aligned} \tag{3.56}$$

and the optimization problem for identification is similar to (3.55) with (Note that (3.49) can be used to obtain (3.57))

$$\theta_P = \begin{cases} [1+c_1, 1+c_1+c_2, \cdots, 1+c_1+\cdots+c_P, 1, c_1, \cdots, c_m, d_1, \cdots, , d_n]^T & \text{for } m \geq P \\ [1+c_1, 1+c_1+c_2, \cdots, 1+c_1+\cdots+c_m, 1, c_1, \cdots, c_m, d_1, \cdots, , d_n]^T & \text{for } m < P \end{cases} \tag{3.57}$$

3.4.1 Results

The results presented in the previous sections are illustrated through a few examples in this section.

Example 3.4.2 Let us consider a process with no exogenous inputs of the form (Since we are dealing with FIR models noise model plays an important role in minimizing the multi-step ahead prediction error variance)

$$y_t = \frac{1}{1 - a_0 q^{-1}} e_t \tag{3.58}$$

where e_t is white with a unit variance then

$$J^{(P)} = \frac{1}{2\pi P} \sum_{j=1}^{P} \int_{-\pi}^{\pi} \left| \hat{W}_j(\omega) \right|^2 \left| \frac{1}{1 - a_0 e^{-i\omega}} \right|^2 d\omega \tag{3.59}$$

assuming that the structure of the noise model is

$$\hat{H} = \frac{1 + cq^{-1}}{1 - q^{-1}}$$

we get

$$\hat{W}_j = \frac{[1 + (1+c)q^{-1} + \cdots + (1+c)q^{-j+1}](1 - q^{-1})}{1 + cq^{-1}}.$$

Using the above the expressions the optimal value of 'c' is plotted as a function of the various prediction horizons in fig.3.1. Clearly, as the prediction horizon increases the estimated multi-step ahead noise model differs from the one-step ahead noise model. Due to the nature of the objective function used in estimating optimal multi-step ahead prediction errors the variance on the multi-step ahead prediction errors is much smaller.

Example 3.4.3 In the previous example we have used a model with no exogenous inputs. Let us now consider a process with exogenous inputs. Consider the process and noise models given below (Clarke *et al.* 1987).

$$G = \frac{0.0077z^2 + 0.0212z + 0.0036}{z^3 - 1.9031z^2 + 1.1514z - 0.2158} \quad H = \frac{z^2}{(z - 0.8)^2} \tag{3.60}$$

A noise model structure of the form

$$\hat{H} = \frac{1}{1 - aq^{-1}} \tag{3.61}$$

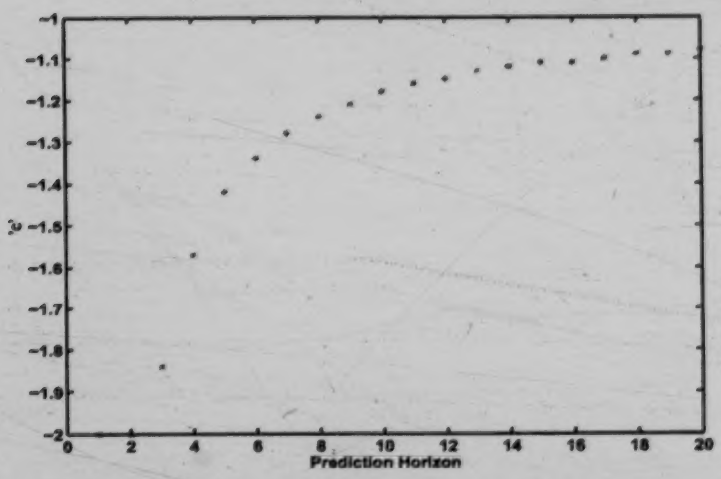


Figure 3.1: The optimal value of the parameter 'c' for various Prediction horizons

is assumed in the simulations. In all the simulations an optimal noise model was identified based on one-step ahead prediction error method and the P-step ahead prediction error method. The data filters, $(\sum_{j=1}^{P} |\hat{W}_j|^2)^{1/2}$, obtained using the traditional one-step ahead error minimization and the MRI method for various prediction horizons are plotted in fig.3.2. Clearly, for $P = 1$ both the methods result in the same filter but as the prediction horizon increases the optimal MRI filter is significantly different from the one-step ahead prediction error filter. The MRI filter provides higher weighting in the low frequency region unlike the traditional identification filters. This is to be expected due to the fact that as the prediction horizon increases the low frequency components need to be emphasized for better predictions. In fact, it can be proved that the weighting becomes uniform as the prediction horizon tends to infinity (Gopaluni *et al.* 2002*a*).

The "optimal" noise pole for the structure used in modelling as the prediction horizon changes is shown in fig.3.3. At $P = 1$ both methods are equivalent but as the prediction horizon increases the MRI noise pole is significantly different. The optimal MRI noise pole seems to converge to a particular value after a certain prediction horizon. In other words, as the prediction horizon tends to infinity (or is very large) the effect of any changes in noise pole become less and less significant. As expected, the noise pole seems to converge to a value that ensures higher weighting in the low frequency region. It is important to emphasize that this behavior of the noise pole depends on the true noise model. It is possible to show that for noise models with monotonically decreasing impulse response coefficients this is true in general (Gopaluni *et al.* 2003*g*). The ratio

Table 3.1: Ratio

P	1	2	3	4	5	6	7	8	9	10
Ratio	1.0000	1.0057	1.0305	0.9467	0.8897	0.8726	0.8609	0.8440	0.8027	0.7154

between the sum squared errors of P-step ahead predictions based on the two methods is plotted in figs.3.4 and 3.5. When $P = 1$, the MRI method and the one-step ahead method are similar and therefore the ratio is unity and it rapidly decreases with the increase in prediction horizon. In other words, the performance of the one-step ahead prediction error model deteriorates when compared with the performance of the P-step ahead prediction error model as we increase the prediction horizon.

Now it is possible to design model predictive controllers based on the different models obtained through open loop identification experiments. For a particular model a corresponding multi-step ahead predictor based on that model is used in implementing MPC algorithm. It is possible to evaluate the designed performance, $J_{des}^{(P)}(t, \mathcal{S})$ for a model $\mathcal{S}$, at each sample instant. The average ratio (averaged over time) between $|J_{ach}^{(P)}(t, \mathcal{S}_1) - J_{des}^{(P)}(t, \mathcal{S}_1)|$ and $|J_{ach}^{(P)}(t, \mathcal{S}_2) - J_{des}^{(P)}(t, \mathcal{S}_2)|$ is shown in Table.3.1. Since the ratio is less than unity for various prediction horizons, the MRI model has performed better at a number of prediction horizons. At $P = 2$ and $P = 3$, the ratio goes above unity indicating worse performance with MRI. This is only due to the variance in the estimates. We have, in fact, noticed that the those two values reach unity as the data length goes to infinity but the rate of convergence is very slow. In general, our experience shows that the improvement gained using MRI model for small prediction horizons is negligible compared to the computational effort involved. However, there is generally significant improvement in the designed performance of the controller with MRI model for large prediction horizons.

Example 3.4.4 Similar results can be obtained for MIMO systems. In this example a 2×2 Wood-Berry column is identified using the proposed two step method. The following process and noise models are used to simulate the data

$$G = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \qquad H = \begin{bmatrix} \frac{3.8e^{-8s}}{14.9s+1} \\ \frac{4.9e^{-3s}}{13.2s+1} \end{bmatrix} \tag{3.62}$$

Noise models of the form

$$\hat{H} = \frac{1 - cq^{-1}}{1 - q^{-1}} \tag{3.63}$$

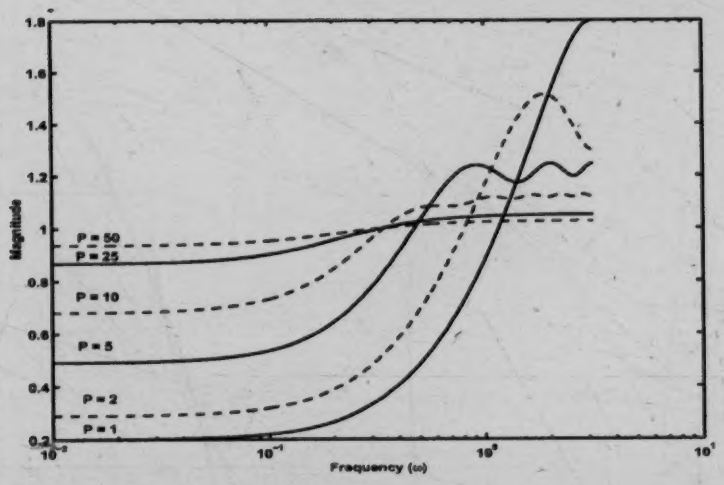


Figure 3.2: The bode plot of the optimal filter $(\sum_{j=1}^{P} |\hat{W}_j|^2)^{1/2}$ for $P = \{1, 2, 5, 25, 50\}$. The dotted line represents the one-step ahead method filter and the solid line is the MRI filter

are then identified for both the outputs. In the first step an unbiased FIR model of the process is identified and in the second step using the unbiased process model two noise models with the above structure are identified for various prediction horizons. The 'c' values of both noise models are plotted for various prediction horizons in fig.3.6.

Example 3.4.5 The usefulness of the two step control relevant identification algorithm is corroborated by experiments on a pilot-scale process shown in fig.3.7. The input/output data is shown in fig.3.8. The experimental set up consists of a cylindrical tank with a steam coil running through it. Water level in the tank is kept constant and the aim is to identify a MPC relevant model between the steam flow rate (u) and the temperature of water (T) at the exit. Using the first step in section 3.4 an FIR model is developed (fig.3.9). An optimal noise model of the form

$$\hat{H} = \frac{1}{1 - aq^{-1}}$$

is then identified for various prediction horizons. figs.3.10 and 3.11 show the noise pole as a function of the prediction horizon (P) and in fig.3.12 the ratio between the loss functions of the P-step ahead predictor using the MPC relevant model and the one-step prediction error model is shown as a function of the prediction horizon. It is clear that there is a significant improvement in the performance of the predictor.

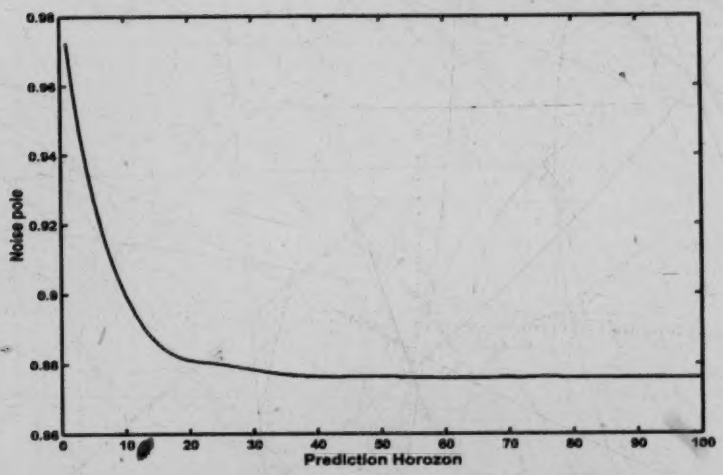


Figure 3.3: Noise pole as function of the prediction horizon

Remark 3.4.1 : In this section we have shown that the optimal identified noise model may differ with the prediction horizon used. If we happen to know the true orders and/or the structures of the noise and the process models then the optimal models are the true models. But in most realistic situations the noise model structure is usually not known. Therefore, it is important to identify an appropriate noise model based on the prediction horizon. Of course, it is also vital to obtain an accurate process model for good closed loop performance but since we are using FIR models that is not an issue here. In case parametric models are used for the process model, bias distribution of the process model determines the performance under closed loop.

3.5 Conclusions and Future work

In this chapter we have discussed in detail various properties of the control relevant predictor obtained by minimizing a multi-step ahead prediction error objective function. It is shown that if the true process and noise model structures are known then either one-step ahead or multi-step ahead prediction error method result in the same optimal model. On the other hand, if the true structures are not known, which is usually the case, the proposed MRI method estimates an optimal multi-step ahead prediction model. The MRI model is shown to provide better designed performance on an average. A few examples are presented to demonstrate these results.

The effect of tuning the noise model on the closed loop performance and on the

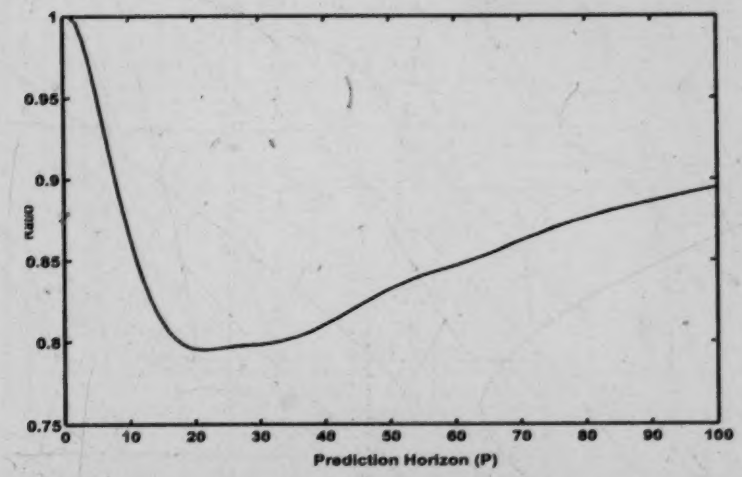


Figure 3.4: Ratio between the prediction error variance obtained using MRI method and the one-step ahead method for various prediction horizons (P)

process model needs to be studied further. An iterative scheme which involves MRI and controller design needs to be developed in order ensure an improvement in achieved closed loop performance. The cross term in (3.4) explains the amount of correlation between the control and identification errors. A thorough understanding of the cross term in both the time and frequency domains is necessary.

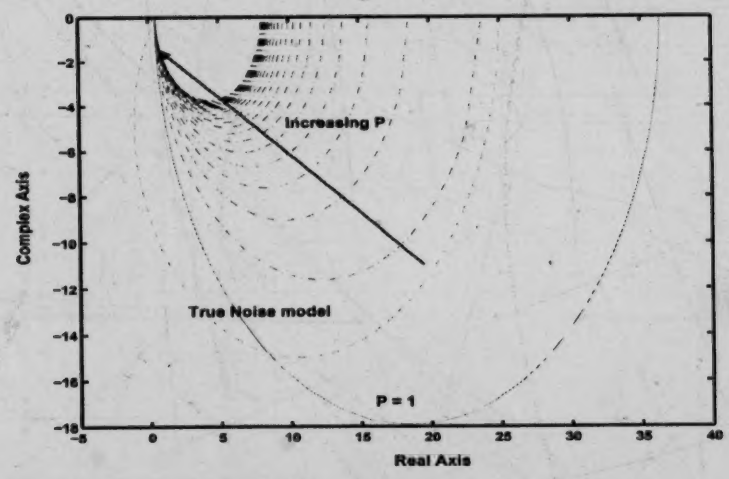


Figure 3.5: Frequency domain comparison of various optimal noise models obtained at all prediction horizons from 1 to 100

3.A Appendix : Proof of Equation (3.3)

$$
\begin{aligned}
J_{ach} &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(r(t) - y(t))^2 \\
&= \lim_{N\to\infty} \frac{1}{PN} \sum_{t=1}^{N} PE(r(t) - y(t))^2 \\
&= \lim_{N\to\infty} \left[\frac{1}{PN} \sum_{t=0}^{N-P} \sum_{j=1}^{P} E(r(t+j) - y(t+j))^2 \right] + \lim_{N\to\infty} \frac{1}{PN} \left[(P-1)E(r(1) - y(1))^2 \right. \\
&\quad + (P-2)E(r(2) - y(2))^2 + \cdots + E(r(P-1) - y(P-1))^2 + \\
&\quad E(r(N-P+1) - y(N-P+1))^2 + \cdots + (P-2)E(r(N-1) - y(N-1))^2 + \\
&\quad \left. (P-1)E(r(N) - y(N))^2 \right] \\
&\approx \lim_{N\to\infty} \left[\frac{1}{PN} \sum_{t=0}^{N-P} \sum_{j=1}^{P} E(r(t+j) - y(t+j))^2 \right] \qquad \text{(3.A -1)}
\end{aligned}
$$

The final equation involving the approximation follows from the assumption that $P << N$.

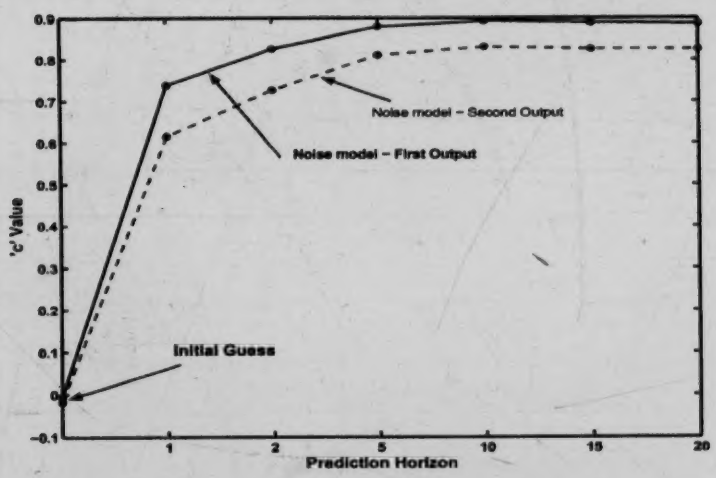


Figure 3.6: Values of 'c' for various prediction horizons

3.B Appendix : Proof of Theorem 3.2.1

It is easy to see that,

$$\begin{aligned} J_{ach}^{(P)}(t,u_f) &= E\left[\sum_{j=1}^{P}(r(t+j)-y(t+j))^2\middle|\mathcal{F}_t\right] \\ &= E\left[\sum_{j=1}^{P}(r(t+j)-\hat{y}_1(t+j))^2\middle|\mathcal{F}_t\right]+E\left[\sum_{j=1}^{P}(\hat{y}_1(t+j)-y(t+j))^2\middle|\mathcal{F}_t\right] \\ &\quad +2E\left[\sum_{j=1}^{P}(r(t+j)-\hat{y}_1(t+j))(\hat{y}_1(t+j)-y(t+j))\middle|\mathcal{F}_t\right] \qquad \text{(3.B -1)} \end{aligned}$$

From the assumptions, we have

$$E\left(\sum_{j=1}^{P}[\hat{y}_1(t+j)-y(t+j)]^2\middle|\mathcal{F}_t\right) \leq E\left(\sum_{j=1}^{P}[\hat{y}_2(t+j)-y(t+j)]^2\middle|\mathcal{F}_t\right)$$

and hence

$$\begin{aligned} J_{ach}^{(P)}(t,u_f) &= E\left[\sum_{j=1}^{P}(r(t+j)-\hat{y}_1(t+j))^2\middle|\mathcal{F}_t\right]+E\left[\sum_{j=1}^{P}(\hat{y}_1(t+j)-y(t+j))^2\middle|\mathcal{F}_t\right] \\ &= J_{des}^{(P)}(t,u_f,\mathcal{S}_1)+E\left[\sum_{j=1}^{P}(\hat{y}_1(t+j)-y(t+j))^2\middle|\mathcal{F}_t\right] \end{aligned}$$

The final result follows by writing the above expression for the second model, $\mathcal{S}_2$.

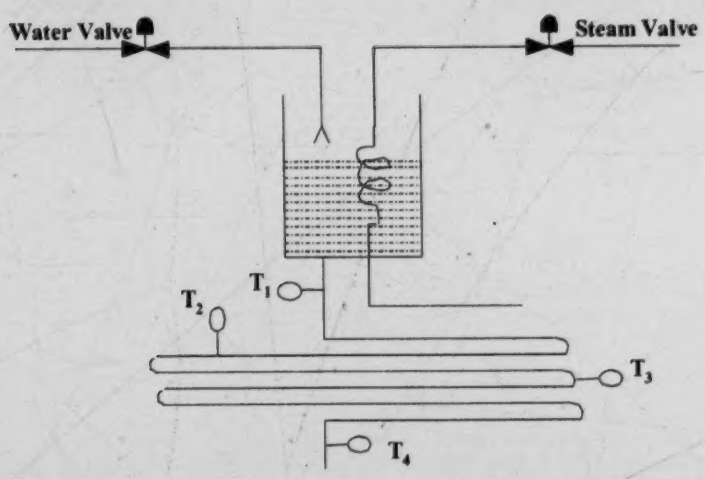


Figure 3.7: A Schematic diagram of the experimental set up

3.C Appendix : MRI as weighted least squares for general models

It is possible to show that the MRI method can be formulated as a weighted least squares problem for a given noise model. The weight being dependent on the impulse response coefficients of the noise model. It is straightforward to show that

$$\begin{aligned} J^{(P)} &= \sum_{j=1}^{P} (y(t+j) - \hat{y}(t+j|t))^2 \\ &= \left[\varepsilon_{t,1}^{p}\right]^2 + \left[\varepsilon_{t,2}^{p} + \hat{g}_1 \varepsilon_{t,1}^{p}\right]^2 + \cdots + \\ &\quad \left[\varepsilon_{t,P}^{p} + \hat{g}_1 \varepsilon_{t,P-1}^{p} + \cdots + \hat{g}_{P-1} \varepsilon_{t,1}^{p}\right]^2 \end{aligned} \tag{3.C -1}$$

Now defining

$$\zeta(t) \triangleq \left[\varepsilon_{t,1}^{p} \quad \cdots \quad \varepsilon_{t,P}^{p}\right]^T \tag{3.C -2}$$

this expression can be written as

$$J \triangleq \zeta^T G^T G \zeta \tag{3.C -3}$$

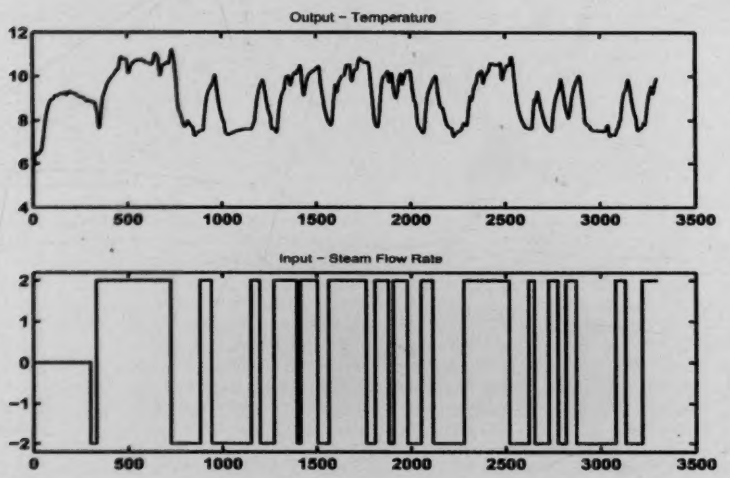


Figure 3.8: Input excitation signal (PRBS) and the corresponding output are shown

where

$$G = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \hat{g}_1 & 1 & 0 \cdots & 0 & \\ \hat{g}_2 & \hat{g}_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{g}_{P-1} & \hat{g}_{P-2} & \cdots & \cdots & 1 \end{bmatrix}_{P \times P} \qquad (3.C\text{-}4)$$

which is clearly a weighted least squares problem with weighting defined by the impulse response coefficients of the noise model. If the noise model is fixed *a priori*, the problem reduces to that of ordinary weighted least squares.

3.D Appendix : Closed loop identification yet another perspective

This is an offshoot of the work on MRI. In closed loop identification there are generally two unknowns *viz.*, an unknown process model and an unknown controller. Therefore, it is necessary to identify the two unknowns separately in order to obtain a reasonable estimate of the process model. This is generally done using the two step method (or joint input-output method) (Van Den Hof and Schrama 1993, Huang and Shah 1997). In this section a method where you don't really need either a two-step method or a pre-whitening filter for closed loop identification is presented. It is always possible to get an unbiased estimate as long as the process model has the correct order (Of course if

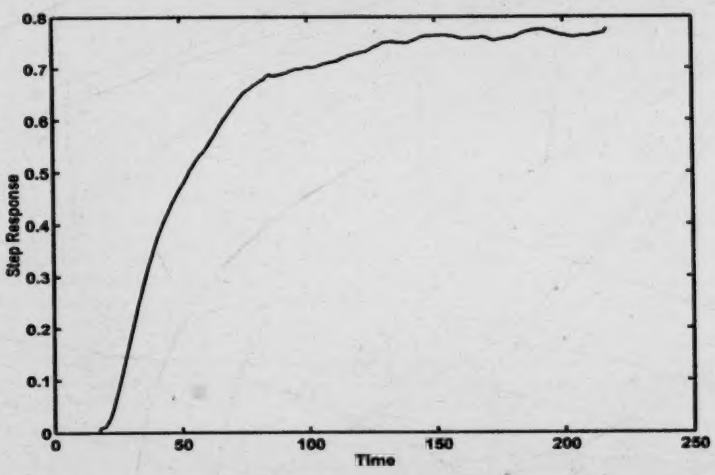


Figure 3.9: The step response of the FIR model between temperature and the steam flow rate

it doesn't, one will have problems. But that is also the case with any other closed loop identification method).

Consider an open loop model

$$y(t) = Gu(t) + He(t) \tag{3.D -1}$$

and a one-step ahead predictor of the form

$$\hat{y}(t+1|t) = \hat{S}\hat{G}u(t+1) + (1-\hat{S})y(t+1) \tag{3.D -2}$$

note that this predictor is different from the optimal predictor

$$\hat{y}(t+1|t) = \hat{H}^{-1}\hat{G}u(t+1) + (1-\hat{H}^{-1})y(t+1) \tag{3.D -3}$$

Now using the predictor in (3.D -2)

$$\bar{E}[(y(t+1) - \hat{y}(t+1|t))^2] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{S}CS|^2|\tilde{G}|^2\Phi_r(\omega) + |HS|^2\Phi_e(\omega)d\omega \tag{3.D -4}$$

the minimum of this expression occurs at $\hat{G} = G$. Therefore, it is always possible to use direct identification to get an unbiased estimate and the trick is to use a special predictor which is not optimal. This predictor is equivalent to fixing the noise model to $\hat{H} = 1/\hat{S}$. Of course, if this optimization problem has more than one solution it is still theoretically possible to get a biased estimate.

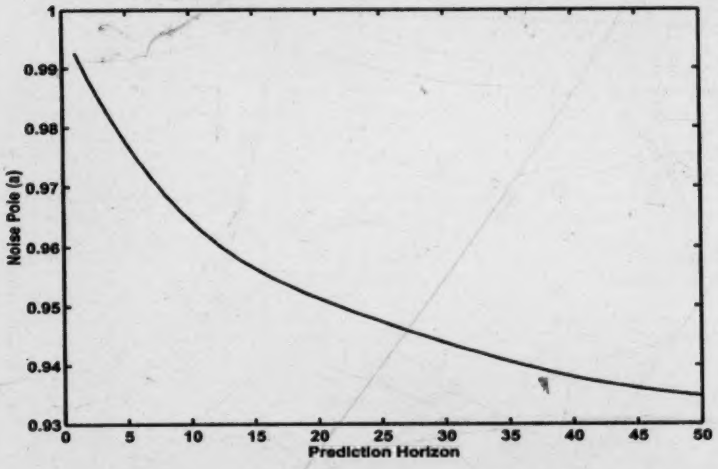


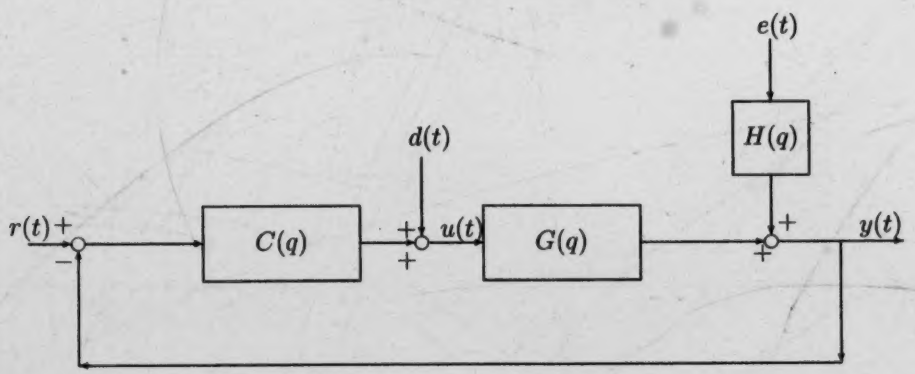
Figure 3.10: Noise pole as a function of Prediction Horizon

3.E Appendix : Nonparametric closed loop identification

Algorithm

A two step method for nonparametric closed loop identification along with the variance expressions for the estimate are presented in this section.
Let us consider a true process model of the form

$$\mathcal{S} \ : \ y(t) = G(q)u(t) + He(t) \tag{3.E -1}$$



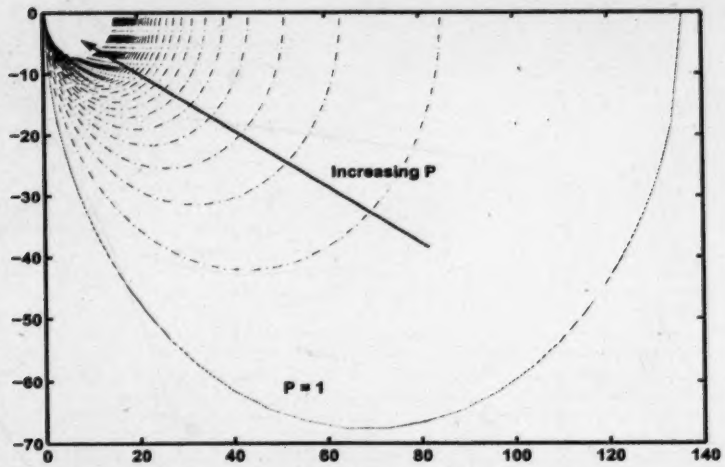


Figure 3.11: Nyquist plot of the noise model as a function of the prediction horizon. All prediction horizons from 1 to 50 are shown.

The closed loop equations are then

$$y(t) = GSd(t) + HSe(t) \quad \text{(3.E -2)}$$

$$u(t) = Sd(t) - SCHe(t) \quad \text{(3.E -3)}$$

The two step closed loop identification involves identifying two open loop systems :

1. Identify the sensitivity function by solving the open loop identification problem between the inputs, u, and the dither signal, d.
2. Filter the dither signal, d, with the sensitivity function and then identify a model between the filtered inputs (u^d) and the outputs (y)

The following two lemmas are useful in deriving expressions for the estimated models using the two step closed loop identification method.

Lemma 3.E .1 *Let $r(t)$ and $w(t)$ be any two quasi-stationary signals and let $S(q)$ be a stable transfer function. Define a new quasi-stationary signal by*

$$r_f(t) = Sr(t) \triangleq \sum_{k=1}^{\infty} s(k)r(t-k) \quad \text{(3.E -4)}$$

then

$$R_{r_f y}(\tau) = \sum_{k=1}^{\infty} s(k)R_{rw}(\tau - k) \quad \text{(3.E -5)}$$

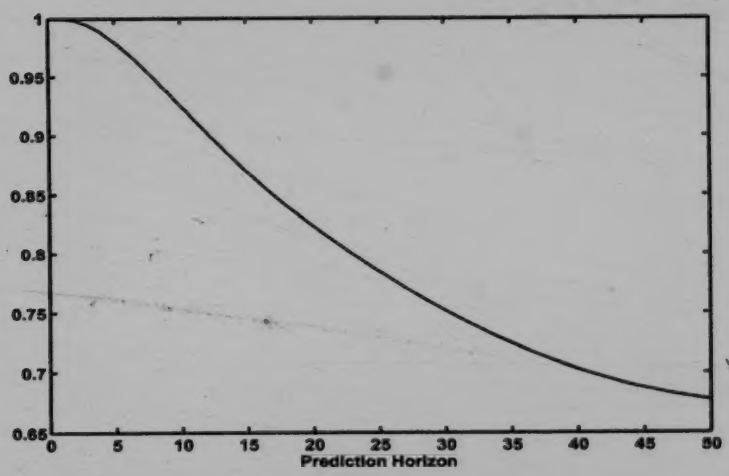


Figure 3.12: Ratio between the loss functions of the P-step ahead predictor using the control relevant model and the one-step prediction error model

Proof:

$$\begin{aligned} R_{r_f w}(\tau) &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} r_f(t) w(t-\tau) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sum_{k=1}^{\infty} s(k) r(t-k) w(t-\tau) \\ &= \sum_{k=1}^{\infty} s(k) \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} r(t-k) w(t-\tau) \\ &= \sum_{k=1}^{\infty} s(k) R_{rw}(\tau - k) \end{aligned} \tag{3.E -6}$$

□

Lemma 3.E .2 *Let $r(t)$ and $w(t)$ be any two quasi-stationary signals and let $S(q)$ be a stable transfer function. Define a new quasi-stationary signal by*

$$r_f(t) = Sr(t) \triangleq \sum_{k=1}^{\infty} s(k) r(t-k) \tag{3.E -7}$$

then

$$\Phi_{r_f w}(\omega) = S(j\omega) \Phi_{rw}(\omega) \tag{3.E -8}$$

Proof:

$$\begin{aligned}
\Phi_{rfw}(\omega) &= \sum_{\tau=-\infty}^{\infty} R_{rfw}(\tau)e^{-i\omega\tau} \\
&= \sum_{\tau=-\infty}^{\infty}\sum_{k=1}^{\infty} s(k)R_{rw}(\tau-k)e^{-i\omega\tau} \\
&= \sum_{k=1}^{\infty} s(k)e^{-i\omega k}\sum_{\tau'=-\infty}^{\infty} R_{rw}(\tau)e^{-i\omega\tau} \\
&= S(j\omega)\Phi_{rw}(\omega) \qquad \text{(3.E -9)}
\end{aligned}$$

□

The transfer function between u and d (sensitivity function) is identified by

$$\hat{S}(j\omega) = \frac{\hat{\Phi}^N_{ud}(\omega)}{\hat{\Phi}^N_{d}(\omega)} \qquad \text{(3.E -10)}$$

and then the process model can be obtained from the expression

$$\begin{aligned}
\hat{G}_N(\omega) &= \frac{\hat{\Phi}^N_{yu^d}(\omega)}{\hat{\Phi}^N_{u^d}} \\
&= \frac{\hat{S}(j\omega)\hat{\Phi}^N_{yd}(\omega)}{\hat{S}(j\omega)\hat{S}(-j\omega)\hat{\Phi}^N_{d}(\omega)} \\
&= \frac{\hat{\Phi}^N_{d}(-\omega)\hat{\Phi}^N_{yd}(\omega)}{\hat{\Phi}^N_{ud}(-\omega)\hat{\Phi}^N_{d}(\omega)} \\
&= \frac{\hat{\Phi}^N_{yd}(\omega)}{\hat{\Phi}^N_{du}(\omega)} \qquad \text{(3.E -11)}
\end{aligned}$$

Therefore a nonparametric model for the process can be estimated by

$$\hat{G}(j\omega) = \frac{\hat{\Phi}^N_{yd}(\omega)}{\hat{\Phi}^N_{du}(\omega)} \qquad \text{(3.E -12)}$$

The noise spectrum can similarly be obtained. Consider the errors between the actual output and the simulated output

$$\hat{\nu}(t) = (y(t)\hat{S}^{-1} - \hat{G}d(t)) := (y_f(t) - \hat{G}d(t)) \qquad \text{(3.E -13)}$$

Therefore the noise spectrum can be estimated by (Ljung 1999)

$$\begin{aligned}
\hat{\Phi}^N_{\nu}(\omega) &= \hat{\Phi}^N_{y_f}(\omega) - \frac{|\hat{\Phi}^N_{y_fd}(\omega)|^2}{\hat{\Phi}^N_{u}(\omega)} \\
&= \frac{|\hat{\Phi}^N_{ud}(\omega)|^2}{\hat{\Phi}^N_{d}(\omega)}\hat{\Phi}^N_{y}(\omega) - \left|\frac{\hat{\Phi}^N_{yd}(\omega)}{\hat{\Phi}^N_{ud}(-\omega)}\right|^2 \qquad \text{(3.E -14)}
\end{aligned}$$

4

Bias distribution of MPC relevant identification methods

In this chapter **M**odel **P**redictive **C**ontrol **R**elevant **I**dentification (MRI) method is applied to a general class of PEM models and the effect of bias distribution on the multi-step ahead predictions is studied. Good multi-step ahead predictions are essential for model predictive controllers. Therefore it is important to distribute the bias in such a way that it is compatible with the predictive control objective. This chapter deals with the analysis of use of MRI methods on the bias distribution and its effect on the control loop performance. Unlike, FIR models (which are not parsimonious), PEM models (in many cases) result in certain amount of bias in the process and noise models. In fact, system identification is essentially an exercise in obtaining reduced complexity models. Therefore it is important to distribute the bias in such a way that it is useful for control. A number of results on the convergence and consistency of the MRI method are proved in this chapter. A discussion of MRI filters in the context of T-filter and the resulting robustness is also presented. [1]

[1] A part of this chapter was published in the proceedings of the IFAC world congress, 2002 (Gopaluni *et al.* 2002*a*). A larger version including the analysis in the closed loop will appear in Automatica, 2003 (Gopaluni *et al.* 2003*g*)

4.1 Introduction

In Chapter 3 the usefulness of MPC Relevant Identification in the context of tuning the noise model was discussed. It was assumed that $\hat{G} \in \mathcal{G}$, which is a reasonable assumption considering the fact that FIR models were used. Under these circumstances, noise model tuning plays an important role. One of the disadvantages of using FIR models is that the parameter set is very large (depends on the time constant and the sample time of the process - time constants are, generally, large in chemical processes) and therefore a large data set is required to obtain parameter estimates with reasonably good variance properties. Moreover, parsimonious models like Box-Jenkins model are some times preferred. If such parsimonious models are used it is highly likely that the set of process models over which the search for an optimal model is conducted, does not include the true model. It is, therefore, instructive to study how to distribute bias and variance in these models in order to get the "optimal" multi-step ahead predictions for a fixed model structure.

Identified models are, generally, used for predictions, control, fault detection *etc.* If it is possible to know the true structures of the process and noise models, then clearly, the true models are optimal no matter for what purpose the identified model is used. Asymptotically, true models are also the maximum likelihood estimates. However, in any realistic situation, it is almost impossible to know the true model structures due to various reasons, for example changing disturbance properties, nonlinearities. In such a scenario, the best one can do is to select a model that is suited for the purpose for which it is going to be used. For instance, control requires identified models which are in consonance with the control objectives and for prediction the identified models must have small prediction errors (Gevers 1993).

In this chapter, the objective of identification is to obtain models for model predictive controllers (MPC). Model predictive controllers employ an objective function that makes use of predicted outputs over a finite-time horizon for evaluating the control action at each sample instant. As a result model predictive controllers require good multi-step ahead predictions, as opposed to, one-step ahead predictions (Shook *et al.* 1992, Huang and Wang 1999, Rossiter and Kouvaritakis 2001). If the model orders of the true process and noise models are known *a priori* then the maximum likelihood estimate which gives optimal multi-step ahead predictions is the true process and noise models. However, in practice it is not possible to know the model structures accurately and moreover most of the real processes are nonlinear. Therefore models tuned for multi-step ahead predictions are vital for good closed loop performance when using predictive controllers.

It is important to emphasize that MPC relevant identification (MRI) methods are useful only when certain amount of bias either in the process model or the noise model is expected. Hence the MRI problem reduces to that of optimal distribution of the bias in process and noise models as commensurate with the control objective function.

Huang and Wang (1999) discuss this method in the context of data pre-filtering. They show that the MRI algorithm can be reduced to that of one-step ahead prediction error method by filtering the inputs and outputs with a filter that depends on the noise model. Expressions for closed loop multi-step ahead predictors are also derived in this chapter. They also provide simulation examples to show that a predictive controller under no constraints would perform better with a model developed for multi-step ahead predictions. Though there is no clear proof, as yet, to show that MRI models indeed perform better under closed loop, certain heuristic and quantitative arguments are presented in Shook *et al.* (1992), Gopaluni *et al.* (2003*a*) and in this chapter.

Rossiter and Kouvaritakis (2001), on the other hand, use multiple models for multi-step ahead predictions. They develop an optimal k-step ahead prediction model for each $k \in \{1, 2, \cdots, P\}$ where P is the controller prediction horizon. Then all the P models are simultaneously used for predictions. Even though this method provides "optimal" multi-step ahead predictions for ARX type models, the number of parameters involved in estimating the models can be quite large. In fact, it is well known that the asymptotic parameter variance is proportional to the ratio between the order of the process model and the data length (Ljung 1999). Since, ARX type models are used in their paper the parameter variance is proportional to the number of parameters and therefore a large data set is required to get reasonably good parameter estimates with acceptable uncertainty. Moreover, for multi-input and multi-output processes the number of parameters can be large. Since the accuracy of multi-step ahead predictions depends on the quality of the noise model, the ARX structure for a model is not appropriate. ARX models do not have enough flexibility in parameterizing the noise model.

This chapter discusses the MRI method from the point of view of distributing the bias in the process model. It is common knowledge that the noise model and the input spectrum determine the bias distribution. Hence, a "good" noise model is essential for good predictions. In fact, if there is no noise model (*i.e.*, if the noise is white) then there is no need for MRI. It is known that the MRI methods differ from the traditional one-step ahead prediction error methods in the choice of a filter that depends on the noise model. An analysis of this filter is presented in this chapter.

4.2 Preliminaries

In order to assess the accuracy of an identified model and to evaluate its statistical properties it is necessary to assume certain properties of the true model. In this chapter and the rest of this thesis it is assumed that the true process is

$$\mathbf{S} \; : \; y(t) = G(q)u(t) + H(q)e(t) \tag{4.1}$$

where $e(t)$ is white noise with variance σ_e^2. $u(t)$ represents the input and $y(t)$ is the output. It is assumed that the input is persistently exciting. For the sake of simplicity the process is assumed to be SISO. q represents the forward shift operator. In the rest of this chapter $G(q)$ is referred as the true process model and $H(q)$ as the true noise model. The estimated model consists of an estimated process model represented by $\hat{G}(q,\theta)$ and an estimated noise model, $\hat{H}(q,\theta)$ where θ is the parameter vector. It is also assumed that $\hat{H}$ and its inverse are both stable.

Then the k-step ahead optimal predictor is given by (Ljung 1999)

$$\hat{y}(t+k|t) \;=\; \hat{W}_k\hat{G}u(t+k) + (1-\hat{W}_k)y(t+k)$$

where

$$\begin{aligned} \hat{W}_k &= \hat{F}_k\hat{H}^{-1} \\ \hat{F}_k &= \sum_{i=0}^{k-1}\hat{h}(i)q^{-i} \; ; \qquad \hat{F}_1 = \hat{h}(0) = 1 \end{aligned}$$

and $\hat{h}(i)$ are the impulse response coefficients of $\hat{H}$. It is assumed that all the inputs and the outputs are quasi-stationary signals.The following standard notation is adopted through out this article. Given a stochastic function $f(t)$, define

$$\bar{E}f(t) := \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}Ef(t) \tag{4.2}$$

where E is the expectation operator (expectation is taken only with respect to the noise). It is assumed that the above limit exists wherever the operator $\bar{E}$ is used. This limit must exist for the signals considered in this chapter due to the assumption that they are all quasi-stationary (Ljung 1999).

We also assume that $\hat{G}$ and $\hat{H}$ are independent of the data length by defining the identified models as

$$\hat{G} \;=\; \lim_{N\to\infty}\hat{G}^N \tag{4.3}$$

$$\hat{H} \;=\; \lim_{N\to\infty}\hat{H}^N \tag{4.4}$$

where $\hat{G}^N$ and $\hat{H}^N$ are models identified from a data of length N. In the rest of this chapter all transfer functions with a superscript 'N' denote the identified models from a data set of length 'N'. Transfer function estimates without the superscript are the model estimates as $N \to \infty$. The following additional notation is introduced for convenience.

$$\tilde{G}(q) := G - \hat{G} \quad \text{and} \quad \tilde{H} := H - \hat{H} \tag{4.5}$$

The data collected from identification experiments on the real process are denoted by

$$Z^N = \{u(1), y(1), \cdots, u(N), y(N)\} \tag{4.6}$$

The following additional assumptions are made in the rest of this thesis (Forssell and Ljung 1999)

Assumption 1. The true system $\mathcal{S}$ is causal, linear and time-invariant with p outputs and m inputs and given by

$$y(t) = G(q)u(t) + \nu(t), \qquad \nu(t) = H(q)e(t) \tag{4.7}$$

where $e(t)$ is a zero mean white noise process with covariance matrix Λ, and bounded moments of order $4+\delta$, some $\delta > 0$, and $H(q)$ is an inversely stable, monic filter.

Assumption 2. The input $\{u(t)\}$ is given by

$$u(t) = k(t, y^t, u^{t-1}, r(t)), \tag{4.8}$$

where $y^t = [y(1), \cdots, y(t)]$, etc., and where the reference signal $\{r(t)\}$ is a given quasi-stationary signal, independent of $\{\nu(t)\}$ and k is a given deterministic function such that the closed loop system is exponentially stable. Exponential stability is defined as follows: For each t, s such that $t \geq s$ there exist random variables $\bar{y}_s(t), \bar{u}_s(t)$, independent of r^s and ν^s, but not independent of r^t and ν^t such that

$$\bar{E}\|y(t) - \bar{y}_s(t)\|^4 < C\lambda^{t-s}$$
$$\bar{E}\|u(t) - \bar{u}_s(t)\|^4 < C\lambda^{t-s}$$

for some $C < \infty$, $\lambda < 1$. In addition, k is such that $G_0(q)u(t)$ is independent of $e(s)$ for $s \geq t$.

Clearly, such assumptions can be questionable under real conditions. Nevertheless, they are useful in analysis and give us an approximate idea of various properties of the estimated models under the assumptions. It is in place to quote this sentence from the legendary book by Ljung (1999) to justify these assumptions.

> *" Analysis pertains to assuming certain properties of the true data-generation mechanism and subsequently calculating the resulting properties of the models. Such calculations turn out to be useful and suggestive, even when the underlying assumptions may not be verifiable."*

Before discussing, the effect of the MRI filter on the bias distribution of the process model it is pertinent to justify the minimization of multi-step ahead predictions in the identification step. Ideally, one would like to minimize the achieved closed loop performance directly (Gevers 2002). This can be done by iteratively identifying the process model. In general, these methods need either an explicit relationship between the model and the controller or a structure of the controller (Hjalmarsson *et al.* 1998) - excepting methods based on iterative feedback tuning (IFT). However, the model predictive controllers have high model orders and it is difficult to derive the control law as an explicit function of the model parameters, especially due to the receding horizon nature of these controllers. As a result, the best that can be done at each sample instant is to design an identification algorithm that ensures a control cost function closer to the intended cost function.

Let us define the vectors comprised of true outputs and predicted outputs from different models as follows : At any sample instant, t

$$Y(t) = [y(t+1) \quad y(t+2) \quad \cdots \quad y(t+P)]^T \tag{4.9}$$

$$Y_1(t) = [y_1(t+1|t) \quad y_1(t+2|t) \quad \cdots \quad y_1(t+P|t)]^T \tag{4.10}$$

$$Y_2(t) = [y_2(t+1|t) \quad y_2(t+2|t) \quad \cdots \quad y_2(t+P|t)]^T \tag{4.11}$$

$$R(t) = [r(t+1) \quad r(t+2) \quad \cdots \quad r(t+P)]^T \tag{4.12}$$

where $Y_1(t)$ and $Y_2(t)$ are predicted outputs over a prediction horizon of P steps from two different models (e.g., model 1 from the MRI method and model 2 from any other identification method).

Proposition 4.2.1 *Let us suppose that*

$$\bar{E}\left[Y^T Y_1\right] = \bar{E}\left[\|Y\|_2 \|Y_1\|_2\right] \tag{4.13a}$$

$$\bar{E}\left[Y^T Y_2\right] = \bar{E}\left[\|Y\|_2 \|Y_2\|_2\right] \tag{4.13b}$$

and also assume that

$$\bar{E}\left[\|Y - Y_1\|_2^2\right] \leq \bar{E}\left[\|Y - Y_2\|_2^2\right] \tag{4.14}$$

then

$$\left|\bar{E}\left[\|R - Y\|_2^2\right] - \|R - Y_1\|_2^2\right| \leq \left|\bar{E}\left[\|R - Y\|_2^2\right] - \|R - Y_2\|_2^2\right| \tag{4.15}$$

Proof: Follows directly from the assumptions. Assumption (4.13) implies that the direction of the vectors $Y(t)$,$Y_1(t)$ and $Y_2(t)$ in the 'P' dimensional space, $\mathbb{R}^P$, is the same. Then the inequality (4.15) follows from the fact that the distance between $Y(t)$ and $Y_1(t)$ in the metric space $(\mathbb{R}^P, \|.\|_2)$[2] is smaller than the distance between $Y(t)$ and $Y_2(t)$.

□

Remark 4.2.1 The assumption in (4.13) is justified as long as the estimated models are *reasonably* good. This result proves that under the assumptions in the proposition the average control cost at each sample instant is closer to the true control cost if multi-step ahead prediction errors are minimized in the identification step. This result as well as the arguments given in (Shook *et al.* 1992, Huang and Wang 1999, Rossiter and Kouvaritakis 2001, Gopaluni *et al.* 2003*a*) provide the motivation for analyzing the effect of minimizing multi-step ahead prediction errors on the identified model.

4.3 Bias In Open loop

Under open loop conditions, all the traditional identification methods try to minimize either one-step ahead prediction errors or k-step ahead prediction errors *i.e.*, an objective function of the form

$$V_N^k(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N-k} \varepsilon_k^2(t,\theta) \tag{4.16}$$

where the k-step ahead prediction errors are given by

$$\varepsilon_k(t,\theta) = y(t+k) - \hat{y}(t+k|t,\theta). \tag{4.17}$$

It is straightforward to show that

$$\bar{E}\left[V_N^k(\theta, Z^N)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{W}_k\tilde{G}\right|^2 \Phi_u(\omega) + \left|\hat{W}_k H\right|^2 \Phi_e(\omega) d\omega \tag{4.18}$$

where $\Phi_u(\omega)$ and $\Phi_e(\omega)$ are input and noise spectra respectively. In this case the filter on the bias is $|\hat{W}_k|^2\Phi_u(\omega) := \hat{L}_k(e^{i\omega})\Phi_u(\omega)$. However, predictive controllers minimize an objective function of the form[3](see fig.4.6 for notation)

$$J_{mpc} = \sum_{j=1}^{P} [r(t) - \hat{y}(t+j|t)]^2 . \tag{4.19}$$

[2]There is a slight abuse of notation here. By the metric space $(\mathbb{R}^P, \|.\|_2)$ we actually mean the metric induced by $\|.\|_2$

[3]for the sake of simplicity it is assumed that there are no time delays

Hence, intuitively speaking, an identification algorithm that can minimize the sum of multi-step ahead prediction errors should give better models for predictive controllers (see Shook *et al.* (1992) and Gopaluni *et al.* (2003*a*) for rigorous arguments). The corresponding or commensurate identification objective function, denoted as MRI, is then defined as

$$V_N(\theta, Z^N, P) = \frac{1}{(N-P)P} \sum_{t=1}^{N-P} \sum_{k=1}^{P} \varepsilon_k^2(t, \theta) \quad . \tag{4.20}$$

Note that the objective function depends explicitly on the controller prediction horizon, P. Once again it can be shown that (using Parseval's theorem)

$$\begin{aligned} \bar{E}\left[V_N(\theta, Z^N, P)\right] &:= \bar{V}(\theta, P) \\ &= \sum_{k=1}^{P} \frac{1}{2\pi P} \int_{-\pi}^{-\pi} \left[|\hat{W}_k \tilde{G}|^2 \Phi_u(\omega) + |\hat{W}_k H|^2 \Phi_e(\omega)\right] d\omega \end{aligned} \tag{4.21}$$

Hence under open loop conditions the optimal weighting on the model bias, $\tilde{G}$, must be

$$\boldsymbol{W_b} := \frac{1}{P} \hat{L}(e^{i\omega}) \Phi_u(\omega) := \frac{1}{P} \sum_{k=1}^{P} |\hat{W}_k|^2 \Phi_u(\omega). \tag{4.22}$$

Clearly, the bias distribution depends to the noise filter and the input spectrum. Notice that it is only the product between the filter, $L(e^{i\omega})$, and the input spectrum, Φ_u, that determines the bias distribution. Their individual values are immaterial. Only the properties of the filter $\hat{L}(e^{i\omega})$ are studied in this chapter.

As mentioned earlier, a k-step ahead predictor has a similar filter (see (4.18)). In this case as the prediction horizon increases (*i.e.,* as k increases) the filter tends to unity. Mathematically,

$$\lim_{k \to \infty} \hat{W}_k = \lim_{k \to \infty} \frac{\tilde{F}_k}{\tilde{H}} \to 1. \tag{4.23}$$

or equivalently, for any given $\delta > 0$, there exists an $N(\delta) \in \mathbf{N}$ such that for all $k \geq N(\delta)$

$$|\hat{W}_k(e^{i\omega}) - 1| < \delta \quad \forall \ \omega. \tag{4.24}$$

Consequently, for any $k < P$ and a corresponding δ and for all ω,

$$(1-\delta)^2 \leq \frac{\hat{L}(e^{i\omega}) - \sum_{j=1}^{k} \hat{L}_j(e^{i\omega})}{P-k} \leq (1+\delta)^2 \tag{4.25}$$

Notice that the upper and lower bounds of the above quantity are independent of the prediction horizon, P and the frequency, ω. Hence, the difference between the lower and upper bounds of $\hat{L}(e^{i\omega})$ is $(P-k)(4\delta)$. Now for any $\varepsilon > 0$, it is possible to choose a P and a δ satisfying

$$k < P \leq \left\lfloor k + \frac{\epsilon}{4\delta} \right\rfloor \tag{4.26}$$

$$\delta \leq \frac{\epsilon}{4} \tag{4.27}$$

such that $(P-k)(4\delta) < \varepsilon$. Where $\lfloor . \rfloor$ denotes the largest integer less than its argument. Therefore, for any arbitrarily small ε there exists a prediction horizon such that the magnitude of the filter $\hat{L}(e^{i\omega})$ is uniform up to a constant ε.

For small values of k, the filters, $\hat{L}_k$, generally tend to give more weighting on high frequencies and as k increases, weighting in the low frequency range improves (Ljung 1999).

On the other hand, MPC relevant filter, $\hat{L}$, provides a weighting that is "optimal" in some sense for multi-step ahead predictions by suitably increasing or decreasing the weighting in the low frequency range depending on the prediction horizon and the identified noise model (see fig.4.1). As the prediction horizon tends to infinity, the weighting tends to be approximately uniform at all frequencies. If the noise model is a low pass filter, effect of using the MRI filter in identification is to increase the weighting on the bias term in the low frequency range compared to one-step ahead prediction models. Naturally, this would result in a better model in the low frequency range. In fact, a good model in the low and mid frequency ranges is essential for good predictions and therefore for good control performance. The following result classifies the noise models that result in an increase in the low frequency weighting

Proposition 4.3.1 *The frequency weighting in the low frequency region increases with the prediction horizon if*

$$\hat{h}_k^2 - 2\hat{h}_k \sum_{i=0}^{k} \hat{h}_i < 0 \quad \forall \ k \in \{1, 2, \cdots, P-1\} \tag{4.28}$$

Proof: It is easy to see that (4.28) is true if and only if

$$|\hat{F}_{k+1}(1)|^2 - |\hat{F}_k(1)|^2 > 0 \quad \forall \ k \in \{1, 2, \cdots, P-1\} \tag{4.29}$$

Now, due to (4.29), $\frac{\hat{L}(1,P)}{P}$ increases with P. Since $\frac{\hat{L}(e^{j\omega},P)}{P}$ is a smooth continuous function of ω, the weighting increases in the low frequency range for noise models satisfying (4.28).

□

Corollary 4.3.1 *All noise models with* $\hat{h}_k > 0$ *for all* k *tend to increase the low frequency weighting with increase in* P.

Remark 4.3.1 Most of the commonly encountered noise models are low pass filters satisfying the conditions in the Corollary 4.3.1. For instance, integrating white noise falls under this category. Therefore, generally, MRI identification methods result in an increase in the low frequency range weighting on the model bias term. Henceforth, only noise models that satisfy the conditions in the above corollary are considered.

The drawback of large weighting in the high frequency region and small weighting in the low frequency region when using one-step ahead prediction error methods is overcome by sufficient input excitation in the low frequency region. MRI methods, in theory should perform better even if the input excitation in the low frequency region is not "sufficient" with appropriately chosen noise models.

4.3.1 Fixed/known noise model

In case the noise model is known *a priori* (which is rarely the case) then the MRI problem reduces to that of minimizing a modified form of the objective function

$$\bar{V}_m(\theta, P) := \sum_{k=1}^{P} \frac{1}{2\pi P} \int_{-\pi}^{\pi} \left[|\hat{W}_k \tilde{G}|^2 \Phi_u(\omega) \right] d\omega. \tag{4.30}$$

Traditional identification methods minimize only the first term in the above equation but on the other hand (4.20) shows that it is important to minimize other terms as well to reduce variance in the prediction errors. If G lies in the set, $\mathcal{G}$, of models over which the optimization is done, then the estimated model is going to be same as the true model and therefore there are no incentives in using MRI models. For example, if FIR models are used, $G \in \mathcal{G}$ and therefore the noise model plays a vital role in determining the performance of the predictor and therefore the controller (Gopaluni *et al.* 2003*a*). Nevertheless, it is impractical to assume that $G \in \mathcal{G}$ since reduced complexity models are of interest and that is when MPC relevant identification method helps us in distributing the bias. An example showing the effect of the MRI filter when the noise model is known is presented below.

[3]Note that $\hat{h}_k > 0$ for all k

Example 4.3.1 The following process and noise models are used in the simulations (from Huang and Wang (1999)).

$$\begin{aligned} G &= \frac{0.0077q^{-1} + 0.0212q^{-2} + 0.0036q^{-3}}{1 - 1.9031q^{-1} + 1.1514q^{-2} - 0.2158q^{-3}} \\ H &= \frac{1}{(1 - 0.8q^{-1})}^4 \end{aligned} \tag{4.31}$$

The noise variance is 0.1 and the input signal is a white RBS signal with variance 1. Also, note that the settling time of this process model is about 40 seconds. An estimated model with the following structure is assumed

$$\hat{G} = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 - a_1 q^{-1}}. \tag{4.32}$$

As explained in the previous section, it is easy to see from the fig.4.1 that the filter $L(e^{i\omega})$

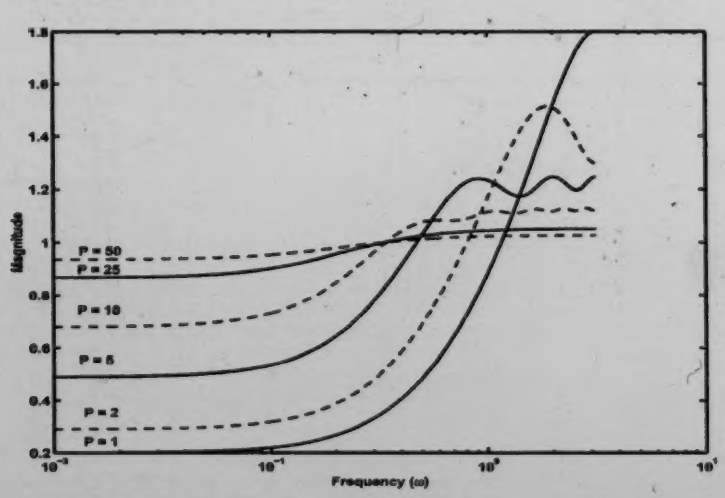


Figure 4.1: Magnitude plot of the bias filter for multi-step ahead predictions - $\frac{L(e^{i\omega})}{P}$

improves the weighting in the low frequency region with increasing prediction horizon. The Bode plots of the estimated models for various prediction horizons are shown in fig.4.2. It is clear that there is a marked improvement of the fit in the low frequency region. Huang and Wang (1999) used an approximation for the MRI filter based on spectral factorization. No approximations for the filter are used. The original objective function is minimized directly, instead of filtering the data. The natural question that arises is: Why use an MRI method? One can instead use OE structure ($\hat{H} = 1$) in identification which naturally is going to result in uniform weighting on the bias

term and therefore provides a good fit in the low frequency region. The MRI method "optimally" adjusts the weighting for multi-step ahead predictions. In other words, it provides a good balance between the low and high frequency ranges for better multi-step ahead predictions.

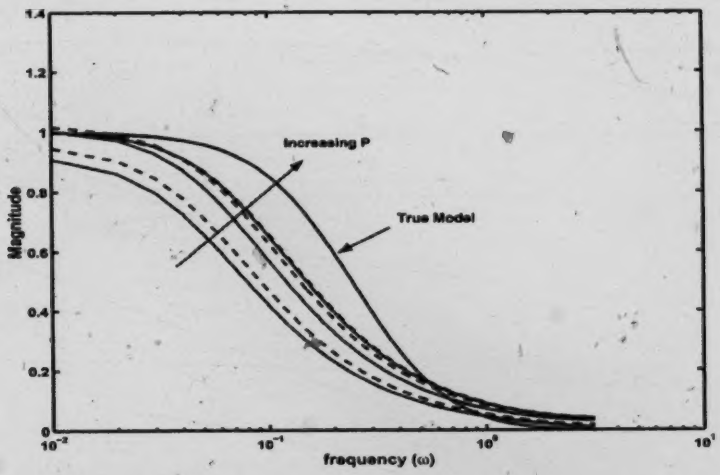


Figure 4.2: Example 1 - Magnitude plot of the estimated first order model for $P = \{1, 2, 5, 10, 25, 50\}$. Magnitude plot of the true process is also shown

4.3.2 Unknown noise model

If the noise model is not known, and if $G \in \mathcal{G}$, the set of models over which the optimization is performed, then the optimization problem can be solved in two steps : 1) Choose an arbitrary noise model *e.g..*, $\hat{H} = 1$ and identify an unbiassed estimate of G. 2) Then identify a noise model by minimizing

$$\sum_{k=1}^{P} \frac{1}{2\pi P} \int_{-\pi}^{-\pi} \left[|\hat{W}_k \hat{H}|^2 \Phi_e(\omega) \right] d\omega. \tag{4.33}$$

In any practical situation, it is impossible to know the true structures of the process and noise models. In such a case, the MRI estimator has to be minimized with respect to both the noise and process models. Clearly, the structure of the process and noise models chosen are going to influence the bias distribution of the estimated process model.

Example 4.3.2 The aim of this example is to show that a bias distribution that is suitable for multi-step ahead predictions is going to be different from the distribution suitable for one-step ahead predictions.

Consider the process model used in the previous example and a new noise model, which is assumed to be unknown with the following structure

$$H = \frac{1}{(1-0.8q^{-1})^2} \tag{4.34}$$

But a noise model with the following structure is assumed to show that the structure of the noise model determines the bias distribution in the process model. All the other parameters are same as those in the previous example.

$$\hat{H} = \frac{1}{1-a_2q^{-1}} \tag{4.35}$$

A plot showing the bias distribution is shown in fig.4.3 and in fig.4.4 the optimal MRI noise pole is plotted against the prediction horizon. Notice that as in the previous example there is a significant improvement in the process model accuracy around low and mid frequency regions at the expense of poor modelling in the high frequency region. The cost function in (4.20) is evaluated for various values of the prediction horizon using the models obtained from the one-step ahead prediction error method and the MRI method. Their ratio is plotted in fig.4.5. There is a significant improvement in the accuracy of the multi-step ahead predictions obtained using the MRI method when compared with the predictions obtained using the one-step ahead prediction error model.

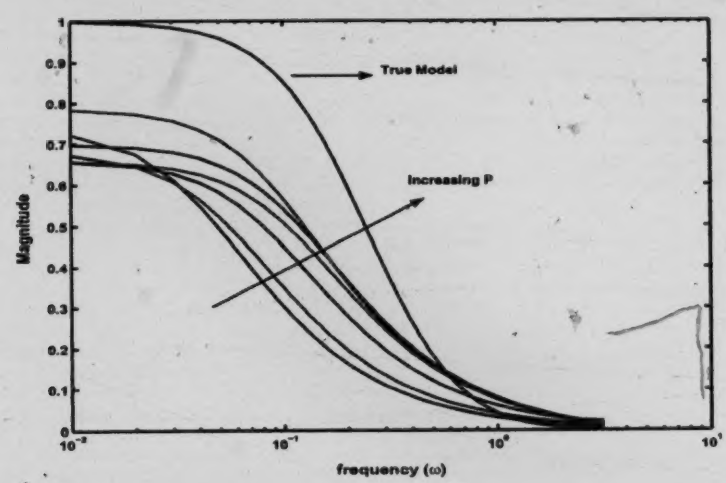


Figure 4.3: Example 2 - The bode magnitude plot of the estimated and true process models for $P \in \{1, 2, 5, 10, 25, 50\}$

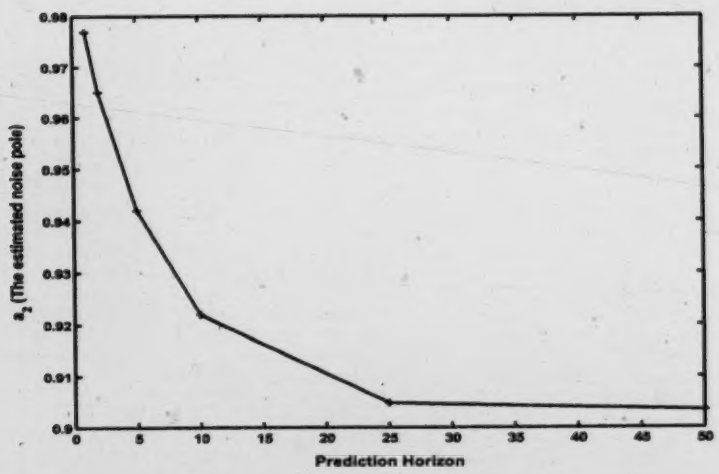


Figure 4.4: The estimated noise pole as a function of prediction horizon

4.4 Bias in closed loop

There is extensive literature on closed loop identification (Forssell and Ljung 1999, MacGregor and Fogal 1995, Huang and Shah 1997) based on minimizing one-step ahead prediction errors. In this chapter closed loop identification based on minimizing multi-step ahead prediction errors is of concern. The expressions for the objective function have to be modified under closed loop to account for the correlation between the input and the noise. Before, proceeding with the expressions for the MRI objective function under closed loop, the following theorem from Forssell and Ljung (1999) is presented for the sake of completeness.

Theorem 4.4.1 (Forssell and Ljung (1999)) *Suppose that the model structure*

$$y(t) = \hat{G}(q,\theta) + \hat{H}(q,\theta)e(t) \tag{4.36}$$

is uniformly stable for $\theta \in D_{\mathcal{M}}$. Assume that the Assumptions 1 & 2 are also satisfied. Then with probability 1 (almost sure convergence) as $N \to \infty$

$$\hat{\theta}_N \to D_c \;=\; \underset{\theta}{\arg\min} \int_{-\pi}^{\pi} \begin{bmatrix} \tilde{G} & \tilde{H} \end{bmatrix} \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Phi_e \end{bmatrix} \begin{bmatrix} \tilde{G} \\ \tilde{H} \end{bmatrix}^* (\hat{H}\Phi_e\hat{H})^{-1} d\omega \tag{4.37}$$

The following proposition shows that the MRI estimates, similarly, converge to a set D_c as $N \to \infty$ with probability 1. Notice that the set D_c is a set of all optimal solutions.

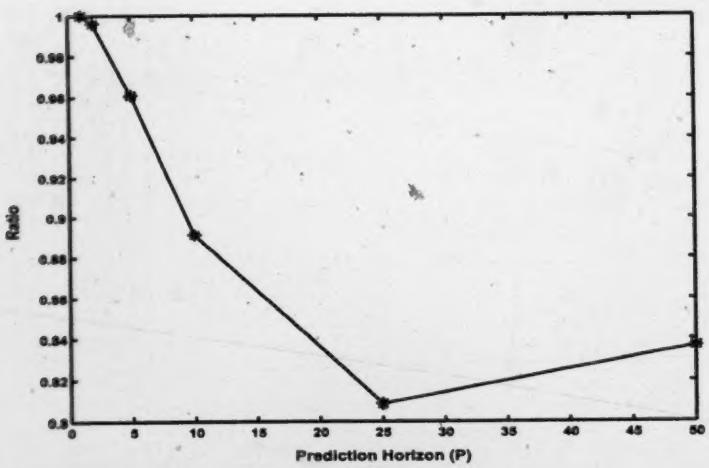


Figure 4.5: The ratio between values of the cost function in (4.20) evaluated using the MRI model and the one-step ahead prediction error model

This is done to account for the non unique solutions. If there is a unique solution then the set has only one element in it.

Proposition 4.4.1 *The MRI estimates of the model in (4.36) almost surely converge to*

$$\begin{aligned}
\hat{\theta}_N \to D_c \;&:=\; \underset{\theta \in D_{\mathcal{M}}}{\arg\min}\; \bar{V}_c(\theta, P) \\
&=\; \underset{\theta \in D_{\mathcal{M}}}{\arg\min}\; \frac{1}{2\pi P} \sum_{k=1}^{P} \int_{-\pi}^{\pi} \begin{bmatrix} \hat{W}_k \tilde{G} & \tilde{W}_k H \end{bmatrix} \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Phi_e \end{bmatrix} \begin{bmatrix} \hat{W}_k \tilde{G} \\ H \tilde{W}_k \end{bmatrix}^* \\
&\quad + \sum_{k=1}^{P} \frac{1}{2\pi P} \int_{-\pi}^{-\pi} |F_k|^2 \Phi_e(\omega) d\omega \qquad (4.38) \\
&=\; \underset{\theta \in D_{\mathcal{M}}}{\arg\min}\; \frac{1}{2\pi P} \sum_{k=1}^{P} \int_{-\pi}^{\pi} \left| \hat{W}_k \tilde{G} \right|^2 |CS|^2 \Phi_r(\omega) + \left| \hat{W}_k H \right|^2 \frac{|S|^2}{|\hat{S}|^2} \Phi_e(\omega) d\omega \qquad (4.39)
\end{aligned}$$

as $N \to \infty$ under Assumptions 1 & 2. Where S and $\hat{S}$ are the true and designed sensitivity functions defined by

$$S := \frac{1}{1 + CG} \qquad \hat{S} := \frac{1}{1 + C\hat{G}}. \qquad (4.40)$$

and Φ_r is the set point spectrum.

Proof: It is easy to see that the k-step ahead prediction errors can be given by

$$\begin{aligned}\hat{\varepsilon}_k(t) &:= \hat{W}_k\tilde{G}u(t) + \hat{W}_kHe(t) \\ &= \hat{W}_k\tilde{G}u(t) + \hat{W}_kHe(t) - W_kHe(t) + F_ke(t) \\ &= \hat{W}_k\tilde{G}u(t) + \left[\frac{\hat{H}_k}{\hat{H}} - \frac{H_k}{H}\right]He(t) + F_ke(t) \end{aligned} \tag{4.41}$$

Now the third term in (4.41) is independent of the second term. As a result, (4.38) follows from (4.41). It is straightforward to prove (4.39) using Parseval's theorem and the assumptions mentioned in this chapter.

□

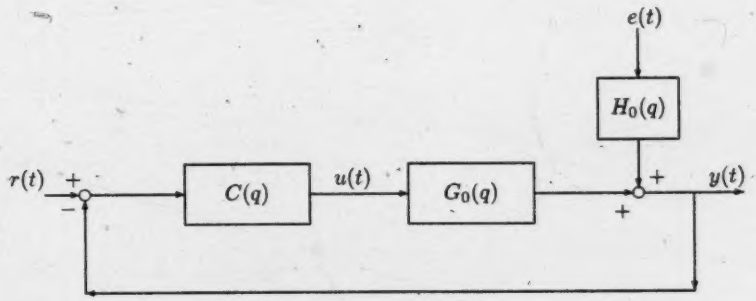


Figure 4.6: Block diagram of a Closed loop system

As is well known there are a number of methods for identifying processes under closed loop conditions. In the following sections, a description of these methods in the context of identifying MRI models is presented

4.4.1 Direct closed loop identification

The direct approach to closed loop identification is similar to the open loop identification method. An input-output model is identified from the data by ignoring the feedback. This method often results in biassed models unless the signal to noise ratio is good. Assuming that the signal to noise ratio is large and that the second term in (4.39) is negligible, the bias weighting on the identified model is

$$\boldsymbol{W_b^C} := \frac{1}{P}\hat{L}_C(e^{i\omega})\Phi_r(\omega) := \frac{1}{P}\sum_{j=1}^{P}|\hat{W}_j|^2|CS|^2\Phi_r(\omega). \tag{4.42}$$

Hence, from (4.23),

$$\lim_{P\to\infty} \boldsymbol{W}_b^{\mathbf{C}}(e^{i\omega}, P) \to |C(e^{i\omega})S(e^{i\omega})|^2\Phi_r \tag{4.43}$$

Example 4.4.1 An IMC controller is designed based on the first order approximation of the original process model in (4.31). A plot of the filter in (4.42) is shown in fig.4.7 for various prediction horizons. Most of the commonly used controllers have an integrator in them and as a result $C(e^{i\omega})$ is large in the low frequency region but on the other hand, a typical sensitivity function is small in the low frequency region. This filter distributes the bias under direct closed loop identification. For other types of closed loop identification *viz.*, indirect and join input-output methods, the bias distribution may depend on different types of filters.

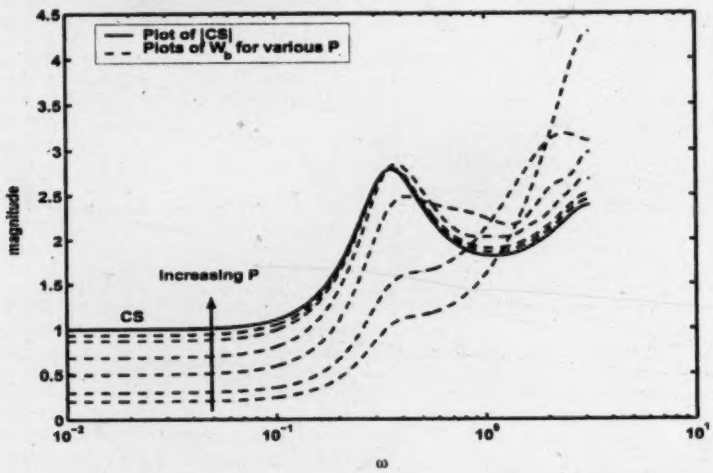


Figure 4.7: The filter, $\boldsymbol{W}_b^{\mathbf{C}}$, in (4.42) is shown for $P \in \{1, 2, 5, 10, 25, 50\}$ with $\Phi_r(\omega) = 1 \quad \forall\ \omega$

As mentioned earlier, the bias weighting in (4.42) is valid only when the signal to noise ratio is large. Otherwise, a term called "bias pull" (Forssell and Ljung 1999) will also have an effect on the bias distribution. The bias pull term will be negligible if the signal to noise ratio is large. The following result separates the effect of the bias pull term and the bias weighting for MRI estimates.

Corollary 4.4.1 *The MRI estimates of the model in (4.36) almost surely converge to*

$$\begin{aligned}\hat{\theta}_N \to D_c \;:=\; & \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \bar{V}_C(\theta, P) \\ =\; & \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \frac{1}{2\pi P}\sum_{k=1}^{P}\int_{-\pi}^{\pi} |\hat{W}_k|^2|\tilde{G} + B_G^k)|^2\Phi_u + |\tilde{W}_k|^2|H|^2\Phi_e^r d\omega \quad (4.44)\end{aligned}$$

as $N \to \infty$ *under Assumptions* 1 *&* 2. *Where* $B_G^k = (W_k/\hat{W}_k - 1)H\Phi_{eu}\Phi_u^{-1}$ *and* $\Phi_e^r = \Phi_e - \Phi_{eu}^2/\Phi_u$.

Proof: The following decomposition of the input and noise spectral matrix is useful in the rest of the proofs in this chapter.

$$\begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Phi_e \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Phi_{eu}\Phi_u^{-1} & I \end{bmatrix}\begin{bmatrix} \Phi_u & 0 \\ 0 & \Phi_e^r \end{bmatrix}\begin{bmatrix} I & \Phi_u^{-1}\Phi_{ue} \\ 0 & I \end{bmatrix} \quad (4.45)$$

The result follows from substituting (4.45) into (4.38) and ignoring the constant term

$$\sum_{k=1}^{P}\frac{1}{2\pi P}\int_{-\pi}^{-\pi}|F_k|^2\Phi_e(\omega)d\omega.$$

□

Remark 4.4.1 Notice that under open loop conditions there is no correlation between the input and the noise. Consequently, the bias pull term disappears. The bias pull term, B_G^k, can be noncausal depending on the controller. Since $\tilde{G}$ is causal, only the causal part of the bias pull term effects the bias distribution. In such a case it is possible to separate the bias pull term into causal and noncausal parts and simplify the above expression further. Denoting the causal part of a transfer function by $[.]_+$ and noncausal transfer function by $[.]_-$ (notation borrowed from Forssell and Ljung (1999)), the expression in (4.44) can be further simplified

$$\begin{aligned}\bar{V}_C(\theta, P) \;=\; & \frac{1}{2\pi P}\sum_{k=1}^{P}\int_{-\pi}^{\pi}|\hat{W}_k|^2|\tilde{G} + B_G^k|^2\Phi_u + |\tilde{W}_k|^2|H|^2\Phi_e^r d\omega \\ =\; & \frac{1}{2\pi P}\sum_{k=1}^{P}\int_{-\pi}^{\pi}|\hat{W}_k|^2|\tilde{G} + B_{G+}^k|^2\Phi_u + |B_{G+}^k|^2\Phi_u + |\tilde{W}_k|^2|H|^2\Phi_e^r d\omega \quad (4.46)\end{aligned}$$

where B_{G+}^k is the causal part of the bias pull term. The cross terms disappear as the integrands are either causal with at least unit delay or noncausal with order greater than or equal to one.

Therefore, the bias in the process model is distributed by the bias pull term and $|\hat{W}_j|^2$. If the true noise model is know *i.e.*, $\hat{H} = H$, then the bias pull disappears from the above equation and therefore the bias weighting is similar to the weighting in the open loop case. On the other hand, if the noise model is unknown it is not possible to achieve the required optimal bias distribution in the process model using direct closed loop identification.

4.4.2 Indirect closed loop identification

In the case of indirect closed loop identification it is assumed that the controller transfer function is known *a priori*. As a result the process model can be estimated by identifying the closed loop transfer function. Let us assume that the following model structure is used

$$\begin{aligned} y(t) &= \frac{\hat{G}(q,\theta)C(q)}{1+\hat{G}(q,\theta)C(q)}r(t) + \hat{H}(q,\theta)e(t) \\ &:= \hat{T}r(t) + \hat{H}e(t) \end{aligned} \tag{4.47}$$

As in the previous case if the filtered multi-step ahead prediction errors are minimized, the following result shows that the estimates converge to a set D_c defined as in the following corollary.

Corollary 4.4.2 *The MRI estimates of the model in (4.36) almost surely converge to*

$$\begin{aligned} \hat{\theta}_N \to D_c &:= \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \bar{V}_C(\theta, P) \\ &= \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \frac{1}{2\pi P}\sum_{j=1}^{P}\int_{-\pi}^{\pi} |\hat{Q}|^2|\hat{W}_j|^2|\tilde{T}|^2\Phi_r + |\hat{Q}|^2|\tilde{W}_j|^2|HS|^2\Phi_e d\omega \\ &= \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \frac{1}{2\pi P}\sum_{j=1}^{P}\int_{-\pi}^{\pi} |\hat{Q}|^2|\hat{W}_j|^2|CS\hat{S}|^2|\tilde{G}|^2\Phi_r + |\hat{Q}|^2|\tilde{W}_j|^2|HS|^2\Phi_e d\omega \end{aligned} \tag{4.48}$$

as $N \to \infty$ under Assumptions 1 & 2. Where $\hat{Q}$ is a filter of user's choice.

Proof: Follows directly from (4.38)

□

Observe that, unlike the one-step ahead minimization problems, filtering the multi-step ahead prediction errors is not equivalent to a different choice of noise model. In order to

get the optimal bias distribution for multi-step ahead predictions, the bias distribution of the process model under closed loop must be same as the distribution under open loop. The bias distribution of the process model for indirect closed loop identification is governed by the weighting

$$W_b^c(\omega) := \frac{1}{P}\sum_{j=1}^{P} |\hat{Q}(e^{i\omega})|^2 |\hat{W}_j(e^{i\omega})|^2 |C(e^{i\omega})S(e^{i\omega})\hat{S}(e^{i\omega})|^2 \Phi_r(\omega) \tag{4.49}$$

Given an *a priori* noise model, comparing (4.49) with (4.22), it should be noted that the optimal filter is $\hat{Q} = 1/CS\hat{S}$ assuming that the input spectrum in the open loop case and the set point spectrum in the closed loop case are same. Clearly, this filter is not realizable due to the presence of S in it. Moreover, if the noise model is not known then it is not possible to obtain the bias weighting that is optimal in the open loop sense.

Since the bias distribution also depends on the input or setpoint spectrum, the only way to achieve the required bias distribution is to choose an appropriate spectrum for the excitation signals. The "optimal" excitation signal, in general, depends on the unknown process model and the controller. Despite this, it is still helpful to compare the optimal excitation spectra for open and closed loop identification methods. If $\Phi_u(\omega)$ is the open loop input excitation spectra, then the corresponding closed loop excitation spectra for indirect identification must be $\Phi_r(\omega) = 1/|CS\hat{S}|^2\Phi_u(\omega)$. Since the magnitude of a typical sensitivity function is small in the low frequency region, more excitation is needed in the low frequency region to offset the effect of estimated and true sensitivity functions on the bias distribution. The MRI filter helps in improving the low frequency weighting.

4.4.3 Joint Input-Output identification

There are different ways of performing joint input-output identification. The basic idea behind these methods is to identify both the controller and the process model or some functions of these two models. Only the two step joint input-output identification method is dealt with in this chapter. The following model structure is used to identify the process model

$$y(t) = \hat{G}(q,\theta)\hat{u}(t) + \hat{H}(q,\theta)e(t) \tag{4.50}$$

where $\hat{u} = \hat{S}r(t)$. As in the other cases it is easy to show that (from (4.39)) the estimated parameters almost surely converge to a particular set.

Corollary 4.4.3 *The MRI estimates of the model in (4.36) almost surely converge to*

$$\begin{aligned}\hat{\theta}_N \to D_c \triangleq & \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \bar{V}_C(\theta, P) \\ = & \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \frac{1}{2\pi P}\sum_{k=1}^{P}\int_{-\pi}^{\pi} |\hat{W}_k|^2\left[|\tilde{G}+B|^2\Phi_{\hat{u}}\right] + |\hat{W}_k|^2\Phi_w - |\hat{W}_k|^2|B|^2 d\omega \quad (4.51) \\ = & \underset{\theta \in D_{\mathcal{M}}}{arg\ min}\ \frac{1}{2\pi P}\sum_{k=1}^{P}\int_{-\pi}^{\pi} |\hat{W}_k|^2|\hat{C}\hat{S}|^2\left[|\tilde{G}+B|^2\Phi_r\right] + |\hat{W}_k|^2\Phi_w - |\hat{W}_k|^2|B|^2 d\omega \\ & \quad (4.52)\end{aligned}$$

as $N \to \infty$ *under Assumptions* 1 *&* 2. *Where* $B = G_0\Phi_{\tilde{u}\hat{u}}/(\Phi_r|\hat{C}\hat{S}|^2)$, $\hat{u}(t) = \hat{C}\hat{S}r(t)$, $\tilde{u}(t) = u(t) - \hat{u}(t)$, $w = G\tilde{u}(t) + He(t)$.

Proof: The result follows from (4.38) and (4.50)

□

Notice that since the identification is done in two steps, in the first step the sensitivity function, $\hat{S}$, is estimated and in the second step multi-step ahead prediction errors based on (4.50) are minimized to obtain the process parameters. In general, these two steps are implemented under a fixed noise model. If the noise model is fixed, then the last term in (4.52) is independent of the parameter vector, θ, and therefore the bias in the process model is distributed by the bias pull term and $\frac{1}{P}\sum_{j=1}^{P}|\hat{W}_j\hat{S}|^2$. For projection methods (Forssell and Ljung 2000*a*), the cross spectrum between $\tilde{u}(t)$ and $\hat{u}(t)$ is zero and therefore the bias pull term disappears. Thus, for projection methods the bias is distributed by the weighting $\frac{1}{P}\sum_{j=1}^{P}|\hat{W}_j\hat{S}|^2$. This filter increases the weighting in the low frequency region. Also, it depends on the accuracy of the sensitivity function obtained in the first step. This is not the optimal weighting on the bias in the open loop sense. Moreover, if the noise model needs to be estimated simultaneously then it is not clear how the bias is distributed due to the dependence of the third term in (4.52) on the bias pull term.

4.4.4 Set point vs dither excitation

The bias distribution also depends on the location of the excitation signal. Typically, the excitation signal is introduced either at the set point or the input (also called dither signal). In the previous section the bias weighting under set point excitation is presented. Table 4.1 compares the bias weighting under both the cases for different closed loop identification methods. Due to the presence of the controller, C, in the bias

Method	Set Point	Dither
Direct	-	-
Indirect	$\frac{\lvert\hat{Q}\rvert^2}{P}\sum_{j=1}^{P}\lvert\hat{W}_j\rvert^2\lvert CS\hat{S}\rvert^2$	$\frac{\lvert\hat{Q}\rvert^2}{P}\sum_{j=1}^{P}\lvert\hat{W}_j\rvert^2\lvert S\hat{S}\rvert^2$
Joint (projection method)	$\frac{1}{P}\sum_{j=1}^{P}\lvert\hat{W}_j\hat{C}\hat{S}\rvert^2$	$\frac{1}{P}\sum_{j=1}^{P}\lvert\hat{W}_j\hat{S}\rvert^2$

Table 4.1: Bias weighting - Set point vs dither excitation

weighting obtained using set point excitation, the improvement in the weighting in the low frequency region may not have a significant impact on the final bias distribution of the identified model. On the other hand, for dither signal excitation, the weighting can improve considerably in the low frequency region.

4.5 Concluding Remarks

The nature of the filter obtained by minimizing multi-step ahead predictions, as opposed to, one-step ahead predictions is discussed in detail. The difference between the two filters lies in improving the weighting on the bias term in the low and mid frequency ranges. MRI method improves the bias weighting in the low frequency region in an "optimal" way taking into account the prediction horizon of the controller. MRI methods showed a significant improvement in the accuracy of multi-step ahead predictions compared to the traditional methods. MRI methods do away with the need for exclusively low frequency excitation. Different closed loop identification methods are also discussed in the context of MPC relevant identification. Further investigation into the effect of MRI filter on the bias distribution under closed loop is needed. The effect of MRI models on the closed loop performance also needs to be studied further.

5

Design of Experiments for MPC relevant identification

As mentioned in the previous chapters experiment design has traditionally been done not in conjunction with the controller that is being used. Experiments have to be designed in such a way that the resulting model is useful for the particular controller that is being used. The bias and variance properties of identified models depend on various factors including the input spectrum. These properties of an estimated model have to be shaped in such a way that the resulting model is commensurate with the controller. This chapter presents a few results on experiment design for Model Predictive Controllers. It is important to minimize multi-step ahead predictions, as opposed to one-step ahead prediction errors, if Model predictive Controllers are used. An optimal weighting on the model error for multi-step ahead prediction errors is derived. Using this weighting, optimal input spectra are derived for the open loop systems[1].

[1] A part of this chapter was published in the proceedings of the American Control Conference, 2002 (Gopaluni *et al.* 2002*b*) and received the **best paper in the session award**. A larger version including the analysis in the closed loop is being prepared for possible journal publication.

5.1 Introduction

The first step in any identification exercise is to design an experiment. Experiment design involves a number of factors such as selecting inputs and outputs, sample time, input spectrum, data length and various other variables that have a bearing on the data collected. In this chapter, we concentrate only on the choice of input spectrum. In many cases, it is too expensive and time-consuming to conduct experiments. Hence, it is very important to choose the right experimental conditions before conducting the experiments. The choice of input naturally depends on the type of model required. We provide certain results on obtaining optimal input spectra for model predictive controllers.

It is well known that Model Predictive Controllers (MPCs) require models which can provide good multi-step ahead predictions, as opposed to, one-step ahead predictions (Shook *et al.* 1992, Huang and Wang 1999, Rossiter and Kouvaritakis 2001). If the structures of the true process model and the noise model are known *a priori* then the maximum likelyhood estimate which gives optimal multi-step ahead predictions is the true process and noise models. However, in practice it is not possible to know model structures and moreover most of the real processes are nonlinear. Therefore models tuned for multi-step ahead predictions are vital for good closed loop performance when using predictive controllers. It is important to emphasize that MPC relevant identification (MRI) methods are useful only when there is certain amount of bias either in the process model or the noise model. Hence the MRI problem reduces to that of distributing the bias and variance errors in process and noise models appropriately.

Huang and Wang (1999) discuss this method in the context of data prefiltering. They show that the MRI algorithm can be reduced to that of one-step ahead prediction error method by filtering the inputs and outputs with a filter that depends on the noise model. Expressions for closed loop multi-step ahead predictors were also derived in this chapter. They also provide simulation examples to show that a predictive controller under no constraints would perform better with a model developed for multi-step ahead predictions. Though there is no clear proof, as yet, to show that MRI models indeed perform better under closed loop, certain heuristic and quantitative arguments are presented in Shook *et al.* (1992) and Gopaluni *et al.* (2003*a*).

Rossiter and Kouvaritakis (2001), on the other hand, use multiple models for multi-step ahead predictions. They develop an optimal k-step ahead prediction model for each $k \in \{1, 2, \cdots, P\}$ where P is the controller prediction horizon. Then all the P models are simultaneously used for predictions. Even though this method provides "optimal"

multi-step ahead predictions for ARX type of models, the number of parameters involved in estimating the models can be unacceptably large.

This chapter discusses the MRI method from the point of view of distributing the bias and variance errors in the process model by choosing an appropriate input spectrum. It is common knowledge that noise model and the input spectrum determine the bias and variance distribution. Unlike traditional one-step ahead prediction error methods, MRI method has a bias weighting that depends, not on the inverse of noise model, but on some function of it. As a result experiment design for good multi-step ahead predictions involves coming up with an optimal input for a particular weighting on the model errors.

Attempts at deriving optimal input spectra are made in Gevers and Ljung (1986), Hjalmarsson *et al.* (1996), Ljung (1999) and Forssell and Ljung (2000*b*) in the context of a variety of controllers. Most of these methods derive an optimal controller and an optimal input spectra by minimizing a closed loop criteria. In this chapter the emphasis is on obtaining an optimal input spectra by minimizing multi-step ahead prediction errors.

5.2 Preliminaries

5.2.1 Assumptions and Notation

In order to assess the accuracy of an identified model and to evaluate its statistical properties it is necessary to assume certain properties of the true model. We are going to assume that the true process is

$$\mathbf{S} \; : \; y(t) = G(q)u(t) + H(q)e(t) \tag{5.1}$$

where $e(t)$ is white noise with variance σ_e^2. $u(t)$ represents the input and $y(t)$ is the output. It is assumed that the input is persistently exciting. For the sake of simplicity the process is assumed to be SISO. q represents the forward shift operator. In the rest of this chapter we refer to $G_0(q)$ as the true process model and to $H_0(q)$ as the true noise model. The estimated model consists of an estimated process model represented by $\hat{G}(q,\theta)$ and an estimated noise model, $\hat{H}(q,\theta)$ where θ is the parameter vector. It is also assumed that $\hat{H}$ and its inverse are both stable. Occasionally, we use the following signal

$$\nu(t) \; = \; H(q)e(t) \tag{5.2}$$

Then the j-step ahead optimal predictor is given by (Ljung 1999)

$$\hat{y}(t+k|t) \; = \; \hat{W}_k\hat{G}u(t+k) + (1-\hat{W}_k)y(t+k)$$

where

$$\begin{aligned} \hat{W}_k &= \hat{F}_k \hat{H}^{-1} \\ \hat{F}_k &= \sum_{i=0}^{k-1} \hat{h}(i) q^{-i} \; ; \qquad \hat{F}_1 = \hat{h}(0) = 1 \end{aligned}$$

and $\hat{h}(i)$ are the impulse response coefficients of $\hat{H}$. It is assumed that all the inputs and the outputs are quasi-stationary signals.The following standard notation is adopted through out this article. Given a stochastic function $f(t)$, we define

$$\bar{E}f(t) \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ef(t) \tag{5.3}$$

where E is the expectation operator (expectation is taken only with respect to the noise). It is assumed that the above limit exists wherever the operator $\bar{E}$ is used. This limit must exist for the signals considered in this chapter due to the assumption that they are all quasi-stationary (Ljung 1999).

We also assume that $\hat{G}$ and $\hat{H}$ are independent of the data length by defining the identified models as

$$\hat{G} = \lim_{N\to\infty} \hat{G}^N \tag{5.4}$$

$$\hat{H} = \lim_{N\to\infty} \hat{H}^N \tag{5.5}$$

where $\hat{G}^N$ and $\hat{H}^N$ are models identified from a data of length N. In the rest of this chapter all transfer functions with a superscript 'N' denote the identified models from a data set of length 'N'. Transfer function estimates without the superscript are the model estimates as $N \to \infty$. Similarly, the optimal estimated parameter vector is denoted by θ_N and its limit as $N \to \infty$ by θ^*.

The following additional notation is introduced for convenience.

$$\tilde{G}(q) := G - \hat{G} \quad \text{and} \quad \tilde{H} := H - \hat{H} \tag{5.6}$$

$$W_j := F_j + H_j \tag{5.7}$$

$$\hat{W}_j := \hat{F}_j + \hat{H}_j \tag{5.8}$$

$$T_0(q) := [G_0(q) \quad H_0(q)] \tag{5.9}$$

and the model from the data under the experimental conditions, $\mathcal{D}$, is denoted by

$$\hat{T}(q) := \left[\hat{G}(q) \quad \hat{H}(q)\right] \tag{5.10}$$

and the error by

$$\tilde{T} \;:=\; \hat{T}(q) - T(q) := \begin{bmatrix} \tilde{G}(q) & \tilde{H}(q) \end{bmatrix} \tag{5.11}$$

The data collected from identification experiments on the real process are denoted by

$$Z^N \;=\; \{u(1), y(1), \cdots, u(N), y(N)\} \tag{5.12}$$

5.2.2 MPC Relevant Identification

Under open loop conditions, all the traditional identification methods try to minimize either one-step ahead prediction errors or k-step ahead prediction errors *i.e.*, an objective function of the form

$$V_N^k(\theta, Z^N) \;=\; \frac{1}{N}\sum_{t=1}^{N} \varepsilon_k^2(t,\theta) \tag{5.13}$$

where the k-step ahead prediction errors are given by

$$\varepsilon_k(t,\theta) \;=\; y(t+k) - \hat{y}(t+k|t,\theta). \tag{5.14}$$

It is straightforward to show that

$$\bar{E}\left[V_N^k(\theta, Z^N)\right] \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{W}_k\tilde{G}\right|^2 \Phi_u(\omega) + \left|\hat{W}_k H\right|^2 \Phi_e(\omega) d\omega \tag{5.15}$$

where $\Phi_u(\omega)$ and $\Phi_e(\omega)$ are input and noise spectra respectively. In this case the filter on the bias is $|\hat{W}_k|^2\Phi_u(\omega) := \hat{L}_k(e^{i\omega})\Phi_u(\omega)$. However, predictive controllers minimize an objective function of the form[2]($r(t)$ is the set point)

$$J_{mpc} \;=\; \sum_{j=1}^{P} \left[r(t) - \hat{y}(t+j|t)\right]^2. \tag{5.16}$$

Hence, intuitively speaking, an identification algorithm that can minimize the sum of multi-step ahead prediction errors should give better models for predictive controllers (see Shook *et al.* (1992), Gopaluni *et al.* (2003*a*) and Gopaluni *et al.* (2003*g*) for rigorous arguments). The MRI objective function is then defined as

$$V_N(\theta, Z^N, P) \;=\; \frac{1}{(N-P)P}\sum_{t=1}^{N-P}\sum_{k=1}^{P} \varepsilon_k^2(t,\theta) \tag{5.17}$$

[2]for the sake of simplicity we have assumed that there are no time delays

Note that the objective function depends explicitly on the controller prediction horizon, P. It is easy to show that (using Parseval's theorem)

$$\begin{aligned} \bar{E}\left[V_N(\theta, Z^N, P)\right] &\triangleq \bar{V}(\theta, P) \\ &= \sum_{k=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} \left[|\hat{W}_k \tilde{G}|^2 \Phi_u(\omega) + |\hat{W}_k H|^2 \Phi_e(\omega)\right] d\omega \end{aligned} \quad (5.18)$$

Hence under open loop conditions the optimal weighting on the model bias, $\tilde{G}$, must be

$$\mathbf{W_b} \triangleq \hat{L}(e^{i\omega})\Phi_u(\omega) \triangleq \sum_{k=1}^{P} |\hat{W}_k|^2 \Phi_u(\omega). \quad (5.19)$$

Clearly, the bias distribution depends to the noise filter and the input spectrum. Notice that it is only the product between the filter, $L(e^{i\omega})$, and the input spectrum, Φ_u, that determines the bias distribution. Their individual values are immaterial. In this chapter expressions for optimal input spectra are derived for open loop systems.

In the following section it is shown that experiments which are informative for one-step ahead prediction error models are also informative for multi-step ahead prediction error models.

5.3 Informative Experiments

In Chapter 2, we have seen the definition of an informative data set. It is important to ensure that the data collected is informative enough before proceeding with identification exercise. If the data is not informative enough, then it may not be possible to distinguish between models in the model set. In the present section we present certain results on the informativeness of the data in the context of MPC relevant identification.

5.3.1 Open loop experiments

Consider a model set of the form

$$\mathcal{M} = \{G(q,\theta), H(q,\theta) | \theta \in D_{\mathcal{M}}\} \quad (5.20)$$

and suppose that θ_1 and θ_2 correspond to two different models. Let $\epsilon_j^p(t,\theta_1)$ and $\epsilon_j^p(t,\theta_2)$ be the j-step ahead prediction errors with the parameter vectors θ_1 and θ_2 respectively. Also define $G_1 = G(q,\theta_1)$, $G_2 = G(q,\theta_2)$, $\Delta G = G_2 - G_1$ and $\Delta W_j = W_j^{(2)} - W_j^{(1)}$ where

all the models corresponding to θ_i have either a subscript or a superscript 'i'. Recall that the j-step ahead prediction errors are given by

$$\begin{aligned} \epsilon_j^p(t,\theta_1) &= W_j^{(1)}[y(t) - G_1 u] \\ \epsilon_j^p(t,\theta_2) &= W_j^{(2)}[y(t) - G_2 u] \end{aligned} \tag{5.21}$$

Then the difference between the j-step ahead prediction errors from the two models can be expressed as

$$\epsilon_j^p(t,\theta_2) - \epsilon_j^p(t,\theta_1) = -W_j^{(1)}\Delta G u(t+j) + \Delta W_j(G - G_2)u(t+j) + \Delta W_j H e(t+j) \tag{5.22}$$

Since the inputs and the noise are uncorrelated,

$$\begin{aligned} \bar{E}\left[\sum_{j=1}^{P}(\Delta\epsilon_j^p)^2\right] = \frac{1}{2\pi}\sum_{j=1}^{P}\int_{-\pi}^{\pi}|W_j^{(1)}|^2 \Bigg[\left|-\Delta G + (G - G_2)\frac{\Delta W_j}{W_j^{(1)}}\right|^2 \Phi_u(\omega) \\ +|\Delta W_j|^2\frac{|H|^2}{|W_j^{(1)}|^2}\Phi_e(\omega)\Bigg] d\omega \end{aligned} \tag{5.23}$$

Assuming that the data are not informative enough with respect to the model set $\mathcal{M}^*$ then

$$\bar{E}\left[\sum_{j=1}^{P}(\Delta\epsilon_j^p)^2\right] = 0 \tag{5.24}$$

even though both ΔG and ΔW_j are not identically zero. Then from (5.24) it is clear that both the terms in the bracket are identically zero. Therefore

$$\Delta W_j = 0 \quad \forall\ j \tag{5.25}$$

which implies that the first term is zero *i.e.*,

$$|\Delta G|^2\Phi_u(\omega) = 0 \tag{5.26}$$

Now if (5.26) implies that $\Delta G = 0$ then it follows that the two models are equal and the data is informative enough. It is in place to observe that this result is similar to the one obtained for one-step ahead prediction errors (Ljung 1999) and therefore any experiment that is informative enough in the traditional sense is also informative enough for the j-step ahead prediction methods. Hence, all the results, in this context, that are valid for traditional identification methods are also valid for the MRI method. We state the following results from Ljung (1999) without proof.

Theorem 5.3.1 *Consider a set $\mathcal{M}^*$ of SISO models given by (5.20) such that the transfer functions $G(z,\theta)$ are rational functions:*

$$G(q,\theta) = \frac{B(q,\theta)}{F(q,\theta)} = \frac{q^{-n_k}(b_1 + b_2 q^{-1} + \cdots + b_{n_b} q^{-n_b+1})}{1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f}} \tag{5.27}$$

Then an open loop experiment with an input that is persistently exciting of order $n_b + n_f$ is sufficiently informative with respect to $\mathcal{M}^*$.

Corollary 5.3.1 *An open loop experiment is informative if the input is persistently exciting.*

5.3.2 Closed loop experiments

Under closed loop conditions, the condition for informativeness in (5.23) can be modified to account for the correlation between the inputs and the noise resulting in,

$$\bar{E}\left[\sum_{j=1}^{P}(\Delta\varepsilon_j^p)^2\right] = \frac{1}{2\pi}\sum_{j=1}^{P}\int_{-\pi}^{\pi}|W_j^{(1)}|^2\left[\left|-\Delta G + (G-G_2)\frac{\Delta W_j}{W_j^{(1)}}\right|^2 |CS|^2\Phi_r(\omega) + \left|\left(\Delta G - (G-G_2)\frac{\Delta W_j}{W_j^{(1)}}\right)CSH + \Delta W_j\frac{H}{W_j^{(1)}}\right|^2\Phi_e(\omega)\right]d\omega \tag{5.28}$$

In the above expression, the coefficients of $\Phi_e(\omega)$ and $\Phi_r(\omega)$ must be identically zero for

$$\bar{E}\left[\sum_{j=1}^{P}(\Delta\varepsilon_j^p)^2\right] = 0. \tag{5.29}$$

Now, the coefficients of both $\Phi_e(\omega)$ and $\Phi_r(\omega)$ are positive, hence both of them must be identically zero, which implies

$$|\Delta W_j|^2 = 0 \quad \forall\ j \tag{5.30}$$

and

$$\Delta G = 0 \tag{5.31}$$

Hence, closed loop data will be informative enough if the set point is persistently exciting.

5.4 Optimal weighting on the model error-open loop

An optimal model for multi-step ahead predictions can be obtained by minimizing

$$\bar{V}(\theta, P) = \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} \left[|\hat{W}_j \tilde{G}|^2 \Phi_u(\omega) + |\hat{W}_j H|^2 \Phi_e(\omega) \right] d\omega. \tag{5.32}$$

Hence under open loop conditions the optimal weighting on the model bias, $\tilde{G}$, must be $\sum_{j=1}^{P} |\hat{W}_j|^2$. Clearly, the bias is distributed according to the noise filter and the input spectrum. The above expression can be rewritten as (using (5.7) and (5.8))

$$\begin{aligned} \bar{V}(\theta, P) &= \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} \left[|\hat{W}_j \tilde{G}|^2 \Phi_u(\omega) + \left| \frac{H_j}{H} - \frac{\hat{H}_j}{\hat{H}} \right|^2 |H|^2 \Phi_e(\omega) \right] d\omega \\ &\quad + \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} |F_j|^2 \Phi_e(\omega) d\omega \\ &:= \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{-\pi} \left[|\hat{W}_j \tilde{G}|^2 \Phi_u(\omega) + \left| W_j - \hat{W}_j \right|^2 |H|^2 \Phi_e(\omega) \right] d\omega + K \end{aligned} \tag{5.33}$$

where K is a constant independent of the model parameters. The idea is to express $\bar{V}(\theta, P)$ as a function of the error variable $\tilde{T}$ by approximating $\hat{W}_j$ around its "true" value W_j. Using first order approximation we get

$$\hat{W}_j = W_j + \left. \frac{d\hat{W}_j}{dH} \right|_H \tilde{H} \tag{5.34}$$

Hence, using (5.33) a new objective function which is useful in designing optimal inputs can be expressed as (using (5.34) and ignoring the constant term, K)

$$J^{(P)}(\hat{T}_N) := \int_{-\pi}^{-\pi} \tilde{T}_N(e^{j\omega}) C \tilde{T}_N^T(e^{-j\omega}) d\omega \tag{5.35}$$

where it is easy to see that the weighting matrix, C, on the error vector, $\tilde{T}$ is given by

$$\begin{aligned} C(e^{j\omega}) &= \begin{bmatrix} \sum_{j=1}^{P} |\hat{W}_j|^2 & 0 \\ 0 & \left| \left(\frac{d\hat{W}_j}{dH} \right)_H H \right|^2 \end{bmatrix} \\ &:= \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \end{aligned} \tag{5.36}$$

This weighting function gives the relative importance of error at different frequencies. At frequencies where the weighting matrix is large, a better model is required. A typical magnitude plot of the MRI filter, $\hat{L}$, is shown in fig.5.1. This filter has a larger weighting in the low and mid frequency regions compared to one-step ahead prediction error method and is optimal for multi-step ahead predictions. Note that the weighting on the model error in (5.35) is a function of this MRI filter. Since we are dealing

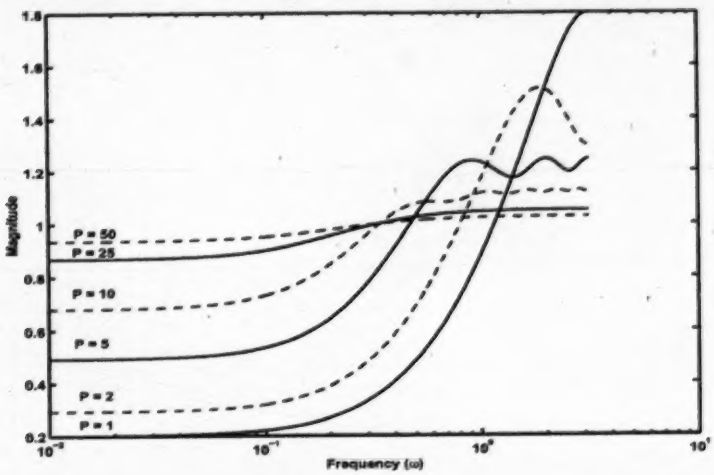


Figure 5.1: Magnitude plot of the bias filter for multi-step ahead predictions - $\frac{L(e^{j\omega})}{P}$ with $\hat{H} = \frac{1}{(1-0.8q^{-1})}$

with open loop systems and there is no correlation between the inputs and the noise, the weighting matrix is diagonal. This weighting matrix will be full under closed loop conditions due to the correlation between the inputs and the output. Notice that the estimated model $\hat{T}_N$ is a random variable *i.e.,* if different sampled data is collected, *i.e.,* a different Z^N, then the model is likely to be different. Therefore,

$$\begin{aligned} J^{(P)} &= E\left[J_N^{(P)}\right] \\ &= E\left[\int_{-\pi}^{-\pi} \tilde{T}_N(e^{j\omega}) C \tilde{T}_N^T(e^{-j\omega}) d\omega\right] \\ &= \int_{-\pi}^{-\pi} E\tilde{T}_N(e^{j\omega}) C \tilde{T}_N^T(e^{-j\omega}) d\omega \\ &= \int_{-\pi}^{-\pi} tr\left[\Pi_N(\omega) C(\omega)\right] \end{aligned} \tag{5.37}$$

where tr denotes trace of a matrix and

$$\Pi_N(\omega) := E\tilde{T}_N(e^{j\omega})\tilde{T}_N^T(e^{-j\omega}). \tag{5.38}$$

There are a number of design criteria that can effect the estimated model and its quality. For example: the structure of the process and noise models, the input spectrum, validation criteria, the identification objective function, the data length etc. All such quantities which have a direct bearing on the model quality are denoted by $\mathcal{D}$. It is clear that $\hat{T}$ and therefore $J^{(P)}$ are functions of $\mathcal{D}$. $J^{(P)}$ can be split into two terms - one that depends on the variance errors and the other that depends on bias errors (see Ljung (1999) for details)- as follows

$$J^{(P)}(\mathcal{D}) = J_V(\mathcal{D}) + J_B(\mathcal{D}) \tag{5.39}$$

with J_V and J_B defined as follows

$$J_V(\mathcal{D}) = \frac{1}{N}\int_{-\pi}^{\pi} tr\left[P(\omega,\mathcal{D})C(\omega)\right]d\omega \tag{5.40}$$

$$J_B(\mathcal{D}) = \int_{-\pi}^{\pi} B(e^{i\omega},\mathcal{D})C(\omega)B^T(e^{-i\omega},\mathcal{D})d\omega \tag{5.41}$$

where as the model order n and the data length N tend to infinity

$$\frac{1}{N}P(\omega,\mathcal{D}) \approx \frac{n}{N}|H(e^{i\omega}|^2\Phi_e(\omega)\begin{bmatrix}\Phi_u(\omega) & \Phi_{ue}(-\omega)\\ \Phi_{ue}(\omega) & \Phi_e(\omega)\end{bmatrix}^{-1} \tag{5.42}$$

and

$$\begin{aligned} B(e^{i\omega},\mathcal{D}) &= \hat{T}(e^{i\omega},\mathcal{D}) - T_0(e^{i\omega}) \\ &= \hat{T}(e^{i\omega},\theta^*(\mathcal{D})) - T_0(e^{i\omega}) \end{aligned} \tag{5.43}$$

The experiment design problem can now be divided into two subproblems of the form

$$\min_{\mathcal{D}} J_V(\mathcal{D}) \tag{5.44}$$

and

$$\min_{\mathcal{D}} J_B(\mathcal{D}) \tag{5.45}$$

Before, presenting the main results of this section we present a lemma that is useful in the proofs of the following theorems.

Lemma 5.4.1 (Ljung (1999)) *Suppose that $A(t)$ and $X(t)$ are two positive functions, then the solution of the optimization problem*

$$\min_{X(t)} \int \frac{A(t)}{X(t)} dt \qquad \text{with constraint } \int X(t)dt \leq K \tag{5.46}$$

is

$$X(t) = \frac{K\sqrt{A(t)}}{\int \sqrt{A(t)}dt} dt \tag{5.47}$$

The following theorem provides the solution to the first subproblem in (5.44)

Theorem 5.4.1 *(a simplified version of the one in Ljung (1999)) Consider the problem in* (5.44) *with respect to the design variable $\mathcal{D} = \Phi_u(\omega)$ under the constraint*

$$\int_{-\pi}^{\pi} \alpha \Phi_u(\omega) d\omega \leq 1 \tag{5.48}$$

for some constant α. Then the optimal solution is given by

$$\Phi_u^{opt} = \mu \sqrt{|H(e^{i\omega})|^2 \Phi_e(\omega) \hat{L}(e^{i\omega})} \tag{5.49}$$

where μ is a constant.

Proof: The proof follows the arguments in Theorem 13.3 of Ljung (1999) very closely. The expression in (5.44) can be simplified to (using (5.42))

$$J_V = \int_{-\pi}^{\pi} \frac{\Phi_e(\omega) C_{11}(\omega) - 2Re[C_{12}(\omega)\Phi_{ue}(-\omega)] + C_{22}(\omega)\Phi_u(\omega)}{\Phi_e(\omega)\Phi_u(\omega) - |\Phi_{ue}(\omega)|^2} \Phi_\nu(\omega) d\omega \tag{5.50}$$

Using (5.36), we can simplify the above equation to

$$J_V = \int_{-\pi}^{\pi} \frac{C_{11}(\omega)}{\Phi_u(\omega)} \Phi_\nu(\omega) d\omega + \int_{-\pi}^{\pi} C_{22} \Phi_\nu(\omega) d\omega \tag{5.51}$$

Now the problem can be reduced to

$$\min_{\Phi_u(\omega)} J_V = \min_{\Phi_u(\omega)} \int_{-\pi}^{\pi} \frac{C_{11}(\omega)}{\Phi_u(\omega)} \Phi_\nu(\omega) d\omega \qquad \text{with } \int_{-\pi}^{\pi} \alpha \Phi_u(\omega) d\omega \leq 1 \tag{5.52}$$

Using Lemma 5.4.1, with $X(\omega) = \alpha \Phi_u(\omega)$ and $A(\omega) = \alpha C_{11}(\omega)\Phi_\nu(\omega)$, we get

$$\Phi_u^{opt}(\omega) = \mu' \frac{1}{\alpha} \sqrt{\alpha C_{11}(\omega)\Phi_\nu(\omega)} \tag{5.53}$$

where μ' is an appropriate constant. Substituting for C_{11} and Φ_ν, we get the result in (5.49).

□

Corollary 5.4.1 *The above problem with a constraint of the form*

$$\int_{-\pi}^{\pi} \alpha\Phi_u(\omega) + \beta\Phi_y d\omega \leq 1 \tag{5.54}$$

has the following solution

$$\Phi_u^{opt} = \mu_1\sqrt{\frac{|H(e^{i\omega})|^2\Phi_e(\omega)\hat{L}(e^{i\omega})}{\alpha+\beta|G|^2}} \tag{5.55}$$

where μ_1 is a constant

Proof: The line of argument is similar to that of Theorem 5.4.1.

□

Remark 5.4.1 The optimal input spectrum is independent of C_{22}. Clearly, the optimal input depends on the true system. In spite of this drawback, the above results give an idea of the shape of the input spectrum that is needed for good multi-step prediction models.

Theorem 5.4.2 *The optimization problem in* (5.45) *with*

$$\theta^* = \underset{\theta}{arg\ min} \int_{-\pi}^{\pi} |\hat{H}|^{-2}|\tilde{G}|^2|Q|^2\Phi_u(\omega) + |\hat{H}|^{-2}|H|^2|Q|^2\Phi_e(\omega) \tag{5.56}$$

where $Q(e^{i\omega})$ is a input-output data filter has the following solution

$$\Phi_u(\omega)|Q(e^{i\omega})|^2 = \mu_3 C_{11}(\omega) \tag{5.57}$$

Proof: The proof follows the same line of argument as that of Theorem 14.1 in Ljung (1999).

□

The solutions of the two subproblems can then be combined to solve the problem of experiment design in (5.39).

Theorem 5.4.3 *The optimization problem*

$$\min_{\mathcal{D}} J^{(P)}(\mathcal{D}) \tag{5.58}$$

with the constraint input in (5.48) *has the following solution*

$$\begin{aligned} \Phi_u(\omega) &= \gamma_1 \sqrt{|H(e^{i\omega})|^2 \Phi_e(\omega) \hat{L}(e^{i\omega})} \\ |Q(e^{i\omega})|^2 &= \gamma_2 \sqrt{\frac{\hat{L}}{|H|^2 \Phi_e(\omega)}} \end{aligned} \tag{5.59}$$

where γ_1 *and* γ_2 *are appropriate constants.*

Proof: Similar to the proof of Theorem 14.2 in Ljung (1999)

□

In almost all the cases the optimal input spectrum is proportional to the MRI filter and the noise spectrum. Therefore, more excitation is needed at frequencies where the noise spectrum is large. For example, if the disturbances entering the process are in the low frequency range then more excitation is needed in the low frequency region. Input spectrum is also proportional to the MRI filter. A typical MRI filter has lower magnitude at low and mid frequency ranges than at high frequencies. Therefore, MRI filter tempers the magnitude of noise spectrum, and as a result the input spectrum, in the low and mid frequency ranges in an optimal way for good multi-step ahead predictions

5.5 Conclusions

A model error criterion based on a weighting relevant to Model Predictive Controllers has been developed in this chapter. This criterion is then minimized with respect to the input spectrum. Few results based on this criteria under different constraints are also presented. It is also shown that an experiment that is informative with respect to a one-step ahead predictor model is informative with respect to the multi-step ahead predictor model. All the results presented in this chapter are valid for open loop systems. Similar results can be derived for closed loop systems also.

Part III

Iterative Identification and Control

6

Iterative Model Tuning for Model Predictive Controllers

In the previous chapters, we have presented a new metric for identification of processes. Building on that idea, in this chapter, we proceed to develop a new iterative algorithm to improve the overall closed loop performance. [1].

6.1 Introduction

System identification and robust control have traditionally been treated as two separate areas with little or no relation between them. For instance, to design a robust controller, frequency domain representation of hard uncertainty bounds are required but on the other hand most of the available identification methods can at the most provide soft uncertainty bounds valid asymptotically as the data length tends to infinity. This line of thinking has changed over the last decade mainly due to the realization that controller design and model identification have to go hand-in-hand in order to design high performance controllers. In other words, there is a need for synergy between identification and control algorithms. Ironically, this is the very same idea based on

[1] A portion of this chapter has been presented at AIChE annual meeting 2001, Reno (Gopaluni *et al.* 2001*b*)

which the dual control problem was proposed four decades ago by Feldbaum (1960-1961). This problem proved to be too difficult to solve due to "the curse of dimensionality" associated with dynamic programming problems. This shift in focus resulted in a number of algorithms that try to design an identification experiment commensurate with the control design method

A number of ideas have emerged in the last decade trying to marry system identification techniques with the robust control methods. Most of these methods can be classified into two broad areas

- Those that attempt to identify uncertainty bounds suitable for robust control
- Those that iteratively achieve improved performance by a succession of identification and control steps.

Most of the early methods concentrated on obtaining suitable error bounds for robust control (these are also called control relevant identification methods) primarily due to the inability of prediction error methods to deliver uncertainty bounds suitable for robust control. Prediction error methods can only provide soft and asymptotic uncertainty bounds. These bounds are not compatible with the uncertainty description required for robust control. As a result a number of control relevant identification methods have been derived (Wahlberg and Ljung 1992, Goodwin *et al.* 1992, Cooley and Lee 1998). A good summary of many of these results can be found in Ninness and Goodwin (1995) and in the references therein. However, using these methods it is not possible to ensure that the nominal model that is used for controller design is the best possible nominal model for high closed loop performance. On top of this, there is no synergy between the controller and the identification algorithm.

The need for synergy between the identification and control lead to the development of Iterative methods called iterative identification and control (Gevers 2002). The idea underlying these methods is to improve the closed loop performance by implementing a succession of identification and robust control steps. A number of such techniques which differ in the identification and robust control criteria have been proposed (Bayard *et al.* 1992, Liu and Skelton 1990, Partanen and Bitmead 1993, Zang *et al.* 1991, Zang *et al.* 1992, de Callafon and Van Den Hof 1997). Most of these methods, assume that the controller transfer function is known. In the case of model predictive controllers, it is difficult to obtain a controller transfer function in terms of the model parameters. Nonetheless, there is no method which can guarantee the convergence of the iterations involved in these methods.

The iterative method by Hjalmarsson *et al.* (1998) involves three identification experiments. They estimate the derivative of the performance objective function at

every stage. The estimates of derivatives are then used to choose the right direction for modifying the controller parameters. This method is independent of the process model. The controller parameters are directly tuned by minimizing the performance objective function.

In this chapter a new iterative algorithm for the simultaneous identification and control using model predictive controllers is developed. Generally, traditional identification methods provide good one-step ahead predictors but on the other hand model predictive controllers need optimal multi-step ahead predictors. As a result the proposed algorithm for iterative identification and control brings in the synergy between the controller and the identification by minimizing multi-step ahead predictions.

6.2 Preliminaries

6.2.1 A motivational example

Traditionally, control and identification are done separately as two independent exercises. In many cases, model identification is a way of coming up with a low order model from a high order process. Consequently, there is always an amount of bias in the identified model. It is important that we shape this bias in such a way that the model results in good closed loop performance. In this section we provide a motivational example to prove this point. Let us consider the following third order model (Clarke *et al.* 1987)

$$G = \frac{0.00768q^{-1} + 0.02123q^{-2} + 0.00357q^{-3}}{1 - 1.9031q^{-1} + 1.1514q^{-2} - 0.2158q^{-3}}$$

with a random-walk disturbance transfer function

$$H = \frac{1}{1 - q^{-1}}$$

To study the effect of various models on the closed loop performance with an MPC controller, the following first order model is used with a range of values for b_2 and a_1

$$\hat{G} = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + a_1 q^{-1}}$$

Fig.6.1 shows a three dimensional plot with the achieved performance (based on the MPC objective function which includes the input weighting) on the z-axis and the parameters b_2 and a_1 on the x and y axis respectively. b_1 was fixed at the least squares estimate for convenience. Fig.6.2 shows the tracking performance of the least squares model in comparison with that of optimal model. The optimal model has a

pole outside the unit circle. Therefore, it is not quite right to use impulse response coefficients in MPC. Nevertheless, there are many other stable models which perform better than the least squares model. It is clear that high performance can be obtained by an appropriate distribution of bias. An iterative identification and control method with the required synergy between the identification and control strategies can improve closed loop performance over that obtained by any other certainty equivalence method with traditional identification strategies.

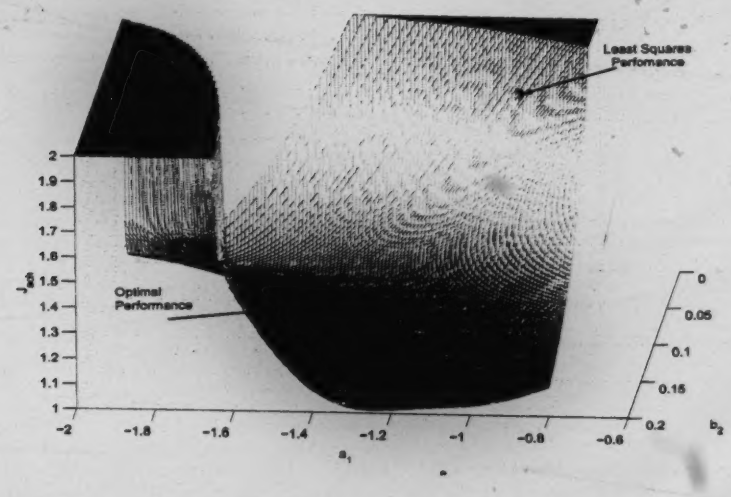


Figure 6.1: A plot showing the closed loop performance as a function of model parameters

Let us consider the following least squares model obtained using the prediction error method

$$\hat{G} = \frac{0.005q^{-1} + 0.0312q^{-2}}{1 - 0.9850q^{-1}} \tag{6.1}$$

and the following model obtained by minimizing multi-step ahead predictions with a prediction horizon of 35 (see Gopaluni *et al.* (2003*a*) for details on obtaining optimal multi-step ahead models)

$$\hat{G} = \frac{0.0001421q^{-1} + 0.1005q^{-2}}{1 - 0.9351q^{-1}}. \tag{6.2}$$

Model predictive controllers designed using the two models with a prediction horizon of 35 are then implemented on the original plant. The following closed loop performance measure is then evaluated for the two controllers.

$$J_{\text{ach}} = \text{sum of control errors} + \text{input weighting} * \text{square of input} \tag{6.3}$$

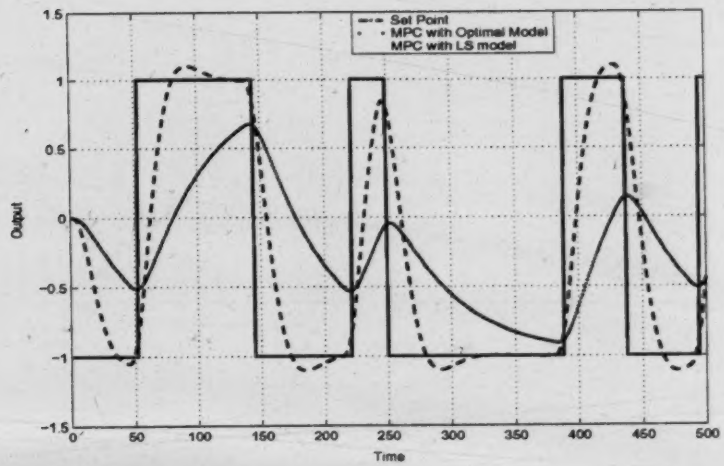


Figure 6.2: Comparison of closed loop responses with the optimal model and the model obtained using prediction error method

The performance measure using the commonly used one-step ahead prediction error model is 419 and that using the optimal model for multi-step ahead predictions is 347. Clearly, there is about 17% improvement in the performance.

The example provided in this section serves as motivation for exploring various issues involved in improving the closed loop performance. There are a number of factors that can have a direct bearing on the performance. For instance, the prediction and control horizons, the input and output weightings, model quality etc. In this chapter we are primarily concerned with the effect of model and its quality on the performance. In other words, we are interested in tuning the model for better performance holding other variables constant.

6.2.2 Assumptions and Notation

In order to assess the accuracy of an identified model and to evaluate its statistical properties it is necessary to assume certain properties of the true model. We are going to assume that the true process is

$$\mathbf{S} \ : \ y(t) = G(q)u(t) + H(q)e(t) \tag{6.4}$$

where $e(t)$ is white noise with variance σ_e^2. $u(t)$ represents the input and $y(t)$ is the output. It is assumed that the input is persistently exciting. For the sake of simplicity the process is assumed to be SISO. q represents the forward shift operator. In the rest

of this chapter we refer to $G_0(q)$ as the true process model and to $H_0(q)$ as the true noise model. The estimated model consists of an estimated process model represented by $\hat{G}(q,\theta)$ and an estimated noise model, $\hat{H}(q,\theta)$ where θ is the parameter vector. It is also assumed that $\hat{H}$ and its inverse are both stable.

Then the j-step ahead optimal predictor is given by (Ljung 1999)

$$\hat{y}(t+k|t) = \hat{W}_k\hat{G}u(t+k) + (1-\hat{W}_k)y(t+k)$$

where

$$\begin{aligned}\hat{W}_k &= \hat{F}_k\hat{H}^{-1} \\ \hat{F}_k &= \sum_{i=0}^{k-1}\hat{h}(i)q^{-i}\ ; \qquad \hat{F}_1 = \hat{h}(0) = 1\end{aligned}$$

and $\hat{h}(i)$ are the impulse response coefficients of $\hat{H}$. It is assumed that all the inputs and the outputs are quasi-stationary signals.The following standard notation is adopted through out this article. Given a stochastic function $f(t)$, we define

$$\bar{E}f(t) \triangleq \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}Ef(t) \tag{6.5}$$

where E is the expectation operator (expectation is taken only with respect to the noise). It is assumed that the above limit exists wherever the operator $\bar{E}$ is used. This limit must exist for the signals considered due to the assumption that they are all quasi-stationary (Ljung 1999).

We also assume that $\hat{G}$ and $\hat{H}$ are independent of the data length by defining the identified models as

$$\hat{G} = \lim_{N\to\infty}\hat{G}^N \tag{6.6}$$

$$\hat{H} = \lim_{N\to\infty}\hat{H}^N \tag{6.7}$$

where $\hat{G}^N$ and $\hat{H}^N$ are models identified from a data of length N. In the rest of this chapter all transfer functions with a superscript 'N' denote the identified models from a data set of length 'N'. Transfer function estimates without the superscript are the model estimates as $N \to \infty$. Similarly, the optimal estimated parameter vector is denoted by θ_N and its limit as $N \to \infty$ by θ^*.

The following additional notation is introduced for convenience.

$$\tilde{G}(q) := G - \hat{G} \quad \text{and} \quad \tilde{H} := H - \hat{H} \tag{6.8}$$

$$W_j := F_j + H_j \tag{6.9}$$
$$\hat{W}_j := \hat{F}_j + \hat{H}_j \tag{6.10}$$

$$T_0(q) := [G_0(q) \quad H_0(q)] \tag{6.11}$$

and the model from the data under the experimental conditions, $\mathcal{D}$, is denoted by

$$\hat{T}(q) := \left[\hat{G}(q) \quad \hat{H}(q)\right] \tag{6.12}$$

and the error by

$$\tilde{T} := \hat{T}(q) - T(q) := \left[\tilde{G}(q) \quad \tilde{H}(q)\right] \tag{6.13}$$

The data collected from identification experiments on the real process are denoted by

$$Z^N = \{u(1), y(1), \cdots, u(N), y(N)\} \tag{6.14}$$

As mentioned in the previous section the idea behind the iterative model tuning is to treat the overall closed loop performance as a function of $G(q,\theta)$ and $H(q,\theta)$ and thereby improve the performance by tuning $G(q,\theta)$ and $H(q,\theta)$.

6.2.3 The need for iterative model tuning

Having showed that appropriate bias distribution is vital for good closed loop performance in the previous section, a basis for the new iterative method is developed in this section. Model predictive controllers (MPC) are widely used in the process industry due to their simplicity and ability to handle constrained multivariable processes. Unlike minimum variance controllers or generalized minimum-variance controllers, model predictive controllers minimize an objective function that is nonmyopic. In other words, the controller minimizes the sum of P-step ahead control errors instead of one-step ahead control errors as is normally done. On the other hand many commonly used identification algorithms do not take this fact into account. A new overall identification criterion, which takes into account the nonmyopic nature of the controller, was defined in Shook *et al.* (1991) and further explored in the context of Generalized Predictive Control (GPC) in Shook *et al.* (1992). This objective function is control relevant in the sense that it minimizes the P-step ahead prediction errors. The same method has been extended to FIR models in Patwardhan *et al.* (1999*a*) and to PEM models in Huang and Wang (1999).

The ultimate objective of any controller is to regulate and keep the output at the setpoint. Many controllers measure the quality of control in terms of the variance of the control error $(r(t) - y(t))$ (where $r(t)$ is the setpoint and y(t) is the controlled output). Hence, the true or the measured output values have to be used in the control objective function instead of predicted values (certainty equivalence method). In other words, at time t, the overall objective function for MPC must be given by[2]

$$J_{MPC} = \sum_{j=1}^{P} (r(t+j) - y(t+j))^2 \tag{6.15}$$

Obviously, $y(t+j)$ is unknown at any time t but that is the objective function that we are interested in minimizing in an MPC. Ideally, we would like to minimize the expected value of the above objective function with respect to the parameters and the control action simultaneously. However, it leads to a nonlinear dynamic optimization problem which is very difficult to solve. Instead a certainty equivalent formulation was proposed by Shook *et al.* (1991). In this approach, the standard objective function of a model based predictive controller in (6.15) is rewritten as

$$J_{MPC} = \sum_{j=1}^{P} \left(r(t+j) - \hat{y}(t+j|t) - \varepsilon_j^p(t)\right)^2 \tag{6.16}$$

$$= \sum_{j=1}^{P} \left(r(t+j) - \hat{y}(t+j|t)\right)^2 + \sum_{j=1}^{P} \left(\varepsilon_j^p(t)\right)^2 - 2\sum_{j=1}^{P} \left(r(t+j) - \hat{y}(t+j|t)\right)\left(\varepsilon_j^p(t)\right) \tag{6.17}$$

where $\varepsilon_j^p(t) = y(t+j) - \hat{y}(t+j|t)$ is the j-step ahead prediction error and the second term is minimized for identification. Any approach that minimizes the three terms independent of each other will be suboptimal. On the other hand, optimizing all three terms simultaneously is a difficult problem. Observe that an MPC minimizes the first term and therefore an identification algorithm which minimizes the second term was called a control relevant identification method in Shook *et al.* (1992). The cross term was ignored in order to facilitate theoretical development. In the following paragraphs a different justification for using the control relevant identification method is presented.

Considering that the closed loop performance is measured by the squared sum of control errors, we have the achieved performance defined by (N is the number of data

[2]We have assumed that there is no delay in the process. It is easy to extend it to the case with delay. We have ignored the input weighting term for simplicity. It can be added with minor modifications.

points)

$$\begin{aligned} J_{ach} &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(r(t) - y(t))^2 \\ &= \bar{E}(r(t) - y(t))^2 \end{aligned} \tag{6.18}$$

It is assumed that the above limit exists. Neglecting the tail/end effects of the control errors (it is true since in most cases $P << N$), we have

$$\begin{aligned} J_{ach} &\approx \frac{1}{P}\bar{E}\left[\sum_{j=1}^{P}(r(t+j) - y(t+j))^2\right] \\ &= \lim_{(N-P)\to\infty} \frac{1}{(N-P)P} \sum_{t=1}^{N-P}\sum_{j=1}^{P} E(r(t+j) - y(t+j))^2 \end{aligned} \tag{6.19}$$

Therefore, minimizing (6.18) is equivalent to minimizing (6.19). Now expanding (6.19) and ignoring the factor $1/P$ for the sake of simplicity, we get

$$\begin{aligned} J_{ach} &\approx \bar{E}\sum_{j=1}^{P}\left(r(t+j) - \hat{y}(t+j|t) - \varepsilon_j^p(t)\right)^2 \\ &= \bar{E}\sum_{j=1}^{P}\left(r(t+j) - \hat{y}(t+j|t)\right)^2 + \bar{E}\sum_{j=1}^{P}\left(\varepsilon_j^p(t)\right)^2 \\ &\quad -2\bar{E}\sum_{j=1}^{P}\left(r(t+j) - \hat{y}(t+j|t)\right)\left(\varepsilon_j^p(t)\right) \end{aligned} \tag{6.20}$$

Clearly, any certainty equivalence formulation of model predictive controller minimizes only the first term in the above equation and as a result it is sub-optimal from a global perspective. Therefore any identification algorithm which minimizes the remaining terms can be expected to be "optimal" in some sense. This idea forms the crux of this chapter. In Huang and Wang (1999) the second term is minimized to obtain a control relevant identifier. This is equivalent to designing a filter which is dependent on the noise model (Gopaluni *et al.* 2003*a*). This identifier is referred to as MPC relevant identifier (MRI) in the rest of the chapter. Since it is difficult to minimize all the three terms simultaneously and obtain an "optimal" model and therefore an optimal controller, an iterative scheme is developed using which the closed loop performance can be improved. In the next three sections a number of properties of the MPC relevant identifier (control relevant identifier) in both time and frequency domains are presented.

6.2.4 Interaction between Control and Identification

Ideally, we should minimize the closed loop performance objective function in (6.18) with respect to the model parameters and the process inputs simultaneously to obtain the "optimal" closed loop performance. Since it is a difficult problem to solve either analytically or numerically, traditionally we take recourse to solving it in two steps *i.e.,* identify a model and then use it to implement a controller optimal with respect to the identified model. In other words, the first two terms in (6.20) are minimized independently without regard for the cross term.

The first term provides a measure of the control errors and the second term similarly provides a measure of identification errors. Finally, the cross term explains the synergy required between identification and control. In this section frequency domain expressions for all the three terms are provided to shed more light on the implications of minimizing these three terms separately. A positive correlation between the control and the identification errors ($\varepsilon^c_{t,j}$ and $\varepsilon^p_{t,j}$) is useful in improving the performance.

Controller Errors - Term I

The first term represents the designed performance of the closed loop. This term is expressed in frequency domain below.
Let us define

$$\varepsilon^c_{t,j} \triangleq (r(t+j) - \hat{y}(t+j/t)) \quad \varepsilon^p_{t,j} \triangleq (y(t+j) - \hat{y}(t+j/t))$$

and let

$$J_C \triangleq \bar{E}\sum_{j=1}^{P}\left(\varepsilon^c_{t,j}\right)^2 \quad J^{(P)} \triangleq \bar{E}\sum_{j=1}^{P}\left(\varepsilon^p_{t,j}\right)^2 \quad J_{ID} \triangleq J^{(P)} - \sum_{j=1}^{P}\bar{E}\left[2\varepsilon^c_{t,j}\varepsilon^p_{t,j}\right] \tag{6.21}$$

then

$$J_{ach} \approx \bar{E}\sum_{j=1}^{P}\left[\left(\varepsilon^c_{t,j}\right)^2 + \left(\varepsilon^p_{t,j}\right)^2 - 2\varepsilon^c_{t,j}\varepsilon^p_j\right] \tag{6.22}$$

$$= \frac{P}{2\pi}\int_{-\pi}^{\pi}|S|^2\,\Phi_r(\omega) + |SH|^2\,\Phi_e(\omega)d\omega \tag{6.23}$$

Now J_C can be expressed as

$$J_C = \sum_{j=1}^{P}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1 + \hat{W}_j\tilde{G}_pC\right|^2|S|^2\,\Phi_r(\omega) + \left|\frac{\hat{W}_j}{\hat{S}} - 1\right|^2|S|^2\,|H|^2\,d\omega \tag{6.24}$$

J_C is the designed performance and J_{ach} is the achieved performance under the designed controller in closed loop.

Identification Errors - Term II

$$J^{(P)} = \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[|\hat{W}_j \tilde{G}|^2 \Phi_u(\omega) + |\hat{W}_j H|^2 \Phi_e(\omega) \right] d\omega \tag{6.25}$$

Cross term - Term III

Some times this term is also referred to as designed performance as it depends on the predicted outputs and not on the true outputs. Consider the following expressions for the control and identification errors in frequency domain.

$$\begin{aligned} \varepsilon^c_{t,j} &= \left[(1 + \hat{W}_j \tilde{G}_p C) S \right] r(t+j) + \left[\frac{\hat{W}_j}{\hat{S}} - 1 \right] SHe(t+j) \\ &\triangleq P^c_j r(t+j) + Q^c_j e(t+j) \end{aligned} \tag{6.26}$$

$$\begin{aligned} \varepsilon^p_{t,j} &= \hat{W}_j \tilde{G}_p CS r(t+j) + \frac{\hat{W}_j G_L S}{\hat{S}} e(t+j) \\ &\triangleq P^p_j r(t+j) + Q^p_j e(t+j) \end{aligned} \tag{6.27}$$

Define

$$\varepsilon_{t,j} \triangleq \left[\varepsilon^c_{t,j} \quad \varepsilon^p_{t,j} \right]^T \tag{6.28}$$

Now assuming noise is independent of the set point excitation, following spectrum for the quasi-stationary signal, $\varepsilon_{t,j}$, is obtained [3]

$$\Phi_{\varepsilon_{t,j}}(\omega) = \begin{bmatrix} \Phi_{\varepsilon^c_{t,j}} & \Phi_{\varepsilon^c_{t,j}\varepsilon^p_{t,j}} \\ \Phi_{\varepsilon^p_{t,j}\varepsilon^c_{t,j}} & \Phi_{\varepsilon^p_{t,j}} \end{bmatrix} \tag{6.29}$$

$$= \begin{bmatrix} P^c_j & Q^c_j \\ P^p_j & Q^p_j \end{bmatrix} \begin{bmatrix} \Phi_r & 0 \\ 0 & \Phi_e \end{bmatrix} \begin{bmatrix} P^c_j & Q^c_j \\ P^p_j & Q^p_j \end{bmatrix}^* \tag{6.30}$$

where $(.)^*$ denotes the complex conjugate and $\Phi_{(.)}$ are the corresponding spectra. therefore

$$\Phi_{\varepsilon^p_{t,j}\varepsilon^c_{t,j}} = P^c_j \Phi_r P^{p*}_j + Q^c_j \Phi_e Q^{p*}_j \tag{6.31}$$

[3] Only set point excitation is considered in this chapter. Similar results can be derived for dither signal excitation

from (6.26) and (6.27) we get

$$\begin{aligned}
\bar{E}[\sum_{j=1}^{P} \varepsilon_{t,j}^{c} \varepsilon_{t,j}^{p}] &= \frac{1}{2\pi} \sum_{j=1}^{P} \int_{-\pi}^{\pi} \Phi_{\varepsilon_{t,j}^{p} \varepsilon_{t,j}^{c}} d\omega \\
&= \frac{1}{2\pi} \sum_{j=1}^{P} \int_{-\pi}^{\pi} \left[P_j^c \Phi_r P_j^{p*} + Q_j^c \Phi_e Q_j^{p*} \right] d\omega \\
&= \frac{1}{2\pi} \sum_{j=1}^{P} \int_{-\pi}^{\pi} \left[(1 + \hat{W}_j \tilde{G}_p C) S \right] \Phi_r(\omega) \left[\hat{W}_j \tilde{G}_p C S \right]^* \\
&\quad + \left[\frac{\hat{W}_j}{\hat{S}} - 1 \right] SH \Phi_e(\omega) \left[\frac{\hat{W}_j G_L S}{\hat{S}} \right]^* d\omega \\
&= J^{(P)} + \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{Re} \left[\Phi_r(\omega) \left(\hat{W}_j \tilde{G}_p C S \right)^* - SH \Phi_e(\omega) \left(\frac{\hat{W}_j G_L S}{\hat{S}} \right)^* \right] d\omega
\end{aligned} \tag{6.32}$$

where **Re** denotes the real part of its complex argument. Therefore the new identification objective function ,J_{ID}, is

$$J_{ID} = -J^{(P)} - 2 \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{Re} \left[\Phi_r(\omega) \left(\hat{W}_j \tilde{G}_p C S \right)^* - SH \Phi_e(\omega) \left(\frac{\hat{W}_j G_L S}{\hat{S}} \right)^* \right] d\omega \tag{6.33}$$

If we know the true process and noise models *i.e.*, $\hat{G} = G$ and $\hat{H} = H$ then the objective function takes the value given by

$$- \frac{1}{2\pi} \sum_{j=1}^{P} \int_{-\pi}^{\pi} |F_j|^2 \Phi_e(\omega) d\omega - 2 \sum_{j=1}^{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{Re} \left[SH \Phi_e F_j^* \right] d\omega \tag{6.34}$$

Clearly its value depends on the true process and noise models. Also, note that since the cross term is proportional to $J_{ach} - J_C - J_{ID}$

$$\bar{E}[\sum_{j=1}^{P} \varepsilon_{t,j}^{c} \varepsilon_{t,j}^{p}] = \frac{1}{2} \left[J_{ach} - J_C - J^P \right] \tag{6.35}$$

$$\begin{aligned}
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[P |S|^2 - \sum_{j=1}^{P} \left(\left| 1 + \hat{W}_j \tilde{G} C \right|^2 |S|^2 - \left| \hat{W}_j \tilde{G} C \right|^2 |S|^2 \right) \right] \Phi_r(\omega) d\omega \\
&+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[P |SH|^2 - \sum_{j=1}^{P} \left(\left| \frac{\hat{W}_j}{\hat{S}} - 1 \right|^2 |S|^2 |H|^2 - \left| \frac{\hat{W}_j}{\hat{S}} \right|^2 |S|^2 |H|^2 \right) \right] \Phi_e(\omega) d\omega
\end{aligned} \tag{6.36}$$

Proposition 6.2.1 *If $\Phi_r(\omega)$ and $\Phi_e(\omega)$ in J_{ID} are constant then*

$$J_{ID} = -J^{(P)} - 2\sum_{j=1}^{P}\frac{1}{2\pi}\int_{-\pi}^{\pi}|SH|^2\,\mathbf{Re}\left(\frac{\hat{W}_j}{\hat{S}}\right)^{*}\Phi_e(\omega)d\omega. \tag{6.37}$$

Proof: If R is a stable transfer function with at least unit time delay then

$$R(e^{i\omega}) = \sum_{k=1}^{\infty} r(k)e^{-ik\omega} \tag{6.38}$$

and then

$$\int_{-\pi}^{\pi} R(e^{i\omega})d\omega = 0 \tag{6.39}$$

The result now follows directly from (6.32)

□

Corollary 6.2.1 *If the excitation sequence is constant i.e., $\Phi_r(\omega) = c$, a constant, and if $\Phi_e(\omega)$ is zero at all frequencies (i.e., the case of no noise) then $J_{ID} = -J^{(P)}$.*

Corollary 6.2.2 *Under the assumptions of corollary 6.2.1, we have*

$$J_{ach} = \bar{E}\left[\sum_{j=1}^{P}(\varepsilon_j^c)^2\right] - \bar{E}\left[\sum_{j=1}^{P}(\varepsilon_j^p)^2\right]. \tag{6.40}$$

Remark: The (6.40) clearly shows that the predictions error have a positive effect in improving the closed loop performance but at the same time large prediction errors can result in the deterioration of the control errors. Therefore, it is very important to strike a balance between the control errors and the prediction errors in order to get good close loop performance.

6.3 An algorithm for Iterative model tuning

MPC relevant identification methods are useful in distributing the model bias and variance errors in such a way that the model gives very small prediction errors with the particular predictor that is being used in the controller. Clearly, the controller depends on the predictor and the predictor depends on the controller. Therefore, optimal performance can only be obtained by modelling the process by taking this interaction

between the controller and the model into account. MRI methods distribute the bias and variance errors to account for this synergy. Since the controller and the identifier are interlinked, this process has to be iterative. In this section, we present an algorithm for iterative identification and control (I2C) and prove some of its properties.
Clearly the controller is trying to minimize the first term $\left(\varepsilon^c_{t,j}\right)^2$ over a set of inputs and the identification algorithm should minimize the other two terms. The second term gives a measure of identification errors. The cross term $2\varepsilon^c_{t,j}\varepsilon^p_{t,j}$ has an interesting interpretation. If the cross term is positive, it aids in improving the performance. Obviously, it can not be a very large positive value (if it is, it will result in high performance). If it takes a large positive value, you will be increasing the other positive terms too. This term is responsible for the synergy between control and identification algorithms. Therefore, ideally we should find a compromise between various terms which can give us optimal performance. The idea is to perform iterations between identification and control a few times so that we improve the performance after every iteration.

Algorithm for iterative identification and control:

Step 1: Obtain an initial guess for the model parameters, θ_0 (*e.g.*, via open loop identification)

Step 2: Implement the controller ($u_i(\theta_i)$) obtained with θ_i and collect sufficiently large closed loop data.

Step 3: Minimize $|J_{ID}(\theta, u(\theta_i))|$ with respect to θ to obtain θ_{i+1}. Design a new controller with θ_{i+1} and implement it in closed loop.

Step 4: Repeat steps 3 and 4 until the achieved performance converges or until the closed loop performance is satisfactory.

Note that the performance objective function is a function of θ, the vector of unknown parameters and u, the sequence of input moves. A schematic diagram of the proposed method is presented in fig.6.3

Proposition: *The proposed algorithm results in minimization of an upper bound on the achieved performance, J_{ach} if the iterations converge.*
Proof:

$$\begin{aligned} J_{ach}(\theta, u(\theta)) &\approx J_c(\theta, u(\theta)) + J_{ID}(\theta, u(\theta)) \\ &\leq J_c(\theta, u(\theta)) + \sqrt{[J_{ID}(\theta, u(\theta))]^2} \end{aligned} \tag{6.41}$$

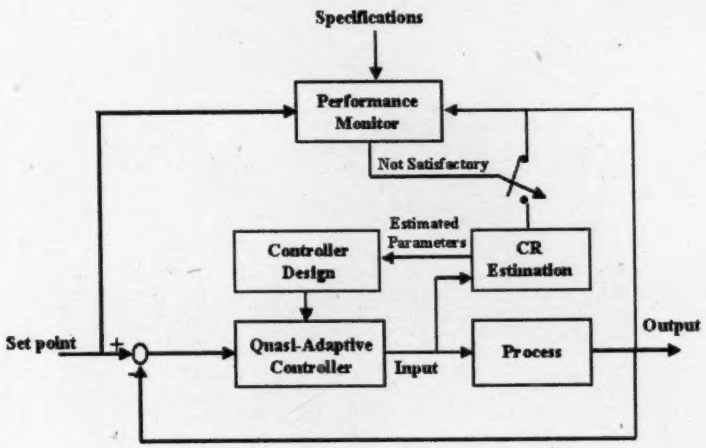


Figure 6.3: A schematic diagram showing the proposed method

The right hand side is an upper bound on the achieved performance as a function of model parameters and the controller. A stationary point of this term occurs at

$$\frac{\partial}{\partial\theta}(J_c(\theta,u(\theta))+\sqrt{[J_{ID}(\theta,u(\theta))]^2})=0$$

which is equivalent to

$$\frac{\partial}{\partial\theta}J_c(\theta,u(\theta)+\frac{1}{2(J_{ID}(\theta,u(\theta)))^{3/2}}\frac{\partial}{\partial\theta}\left[J_{ID}(\theta,u(\theta))\right]^2=0 \tag{6.42}$$

when the iterations converge, we have $u_\infty = u(\theta_\infty)$ where u_∞ and θ_∞ stand for the converged values of the respective variables. Clearly, when the iterations converge both the terms in (6.42) are simultaneously equal to zero. Therefore, $(\theta_\infty, u_\infty)$ is a stationary point. Since both the terms in (6.41) attain their minimum at this stationary point, it must also be the minimum of the upper bound on J_{ach}.

□

Proposition: *The iterations are guaranteed to converge if the following conditions are satisfied*

(a)

$$\left|\frac{\partial^2}{\partial\theta^2}\left[f(u(\theta))\right]\right|\leq\alpha<1 \tag{6.43}$$

where $f(u(\theta))$ is the solution of θ obtained from the equation

$$\frac{\partial}{\partial \theta}[J_{ID}(\theta, u(\theta))]^2 = 0 \tag{6.44}$$

and α is a positive constant

(b) $f(u(\theta))$ has a continuous derivative in some interval around the solution.

Proof: The proof follows from Theorem in Kreyszig (1996) but for the sake of clarity it is reproduced here. Let us assume that we can solve (6.44) for θ, *i.e.*,

$$\theta = f(u(\theta))$$

Now the SIC method is equivalent to performing the following iterations

$$\theta_{k+1} = f(u(\theta_k)) \tag{6.45}$$

where θ_k is the parameter vector at the kth iteration. Let θ_∞ be the solution. Then by the mean value theorem

$$f(u(\theta)) - f(u(\theta_\infty)) = \frac{d}{d\theta} f(u(\theta_m))(\theta - \theta_\infty) \tag{6.46}$$

where θ_m is between θ and θ_∞.
Since $f(u(\theta_\infty)) = \theta_\infty$ and $\theta_1 = f(u(\theta_0))$, $\theta_2 = f(u(\theta_1))$, $\cdots$, we obtain

$$\begin{aligned} |\theta_n - \theta_\infty| = |f(u(\theta_{n-1})) - f(u(\theta_\infty))| &= \frac{d}{d\theta} f(u(\theta_m))|\theta_{n-1} - \theta_\infty| \\ &\leq \alpha|\theta_{n-1} - \theta_\infty| \\ &\leq \alpha^2|\theta_{n-2} - \theta_\infty| \\ &\vdots \\ &\leq \alpha^n|\theta_0 - \theta_\infty| \end{aligned} \tag{6.47}$$

Since $\alpha < 1$, $\alpha^n \to 0$ as $n \to \infty$ and therefore $|\theta_n - \theta_\infty| \to 0$.

□

It is generally, not possible to arrive at an explicit expression for the the condition in (6.43). It is important to note the difference between the identification objective function in the MRI methods and the proposed method. There is an additional cross term in the proposed method which accounts for the synergy between Identification and control. Due to the dependence on this term, proposed method results in an identified model which is suitable for control in the frequency ranges where the set point excitation usually occurs.

6.3.1 Fixed noise model

If the noise model is fixed *a priori* then the MRI objective function would lead to an unbiased FIR model but on the other hand the proposed method can still result in a biased FIR model due to the cross term. In other words, we are trying to adjust the process model for a given noise model to obtain the best possible performance. In order to use the MRI methods effectively, we need to have a good idea about the noise model. If the noise model is not known it is not possible to design the data pre-filter. This drawback of MRI methods is taken care of by adjusting the process model in the proposed method. Including the noise model gives us additional flexibility in minimizing the objective function. In fact using a noise model may give us better performance.

6.4 Example and Results

The new algorithm for identification and control was tried on the example in (4.31). The closed loop performance after each iteration along with the corresponding models are shown in figs.6.4 and 6.5. Fig.6.6 shows the convergence of J_c and J_{ID}. It is important to observe that the performance converges only in the average due to the stochastic nature of the objective function. The objective function itself has a certain variance associated with it and that is the reason for the variations in the achieved performance.

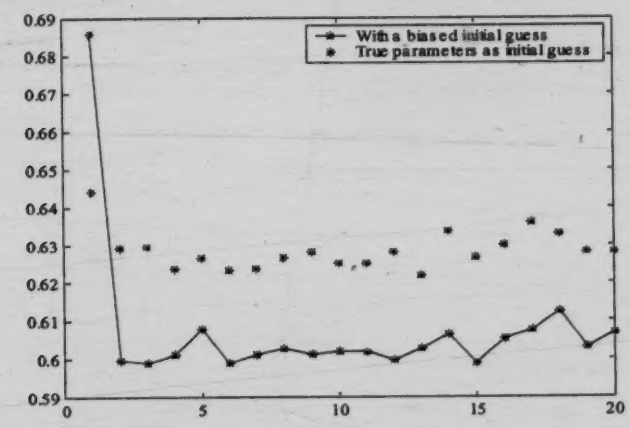


Figure 6.4: Close loop performance vs No. of iterations

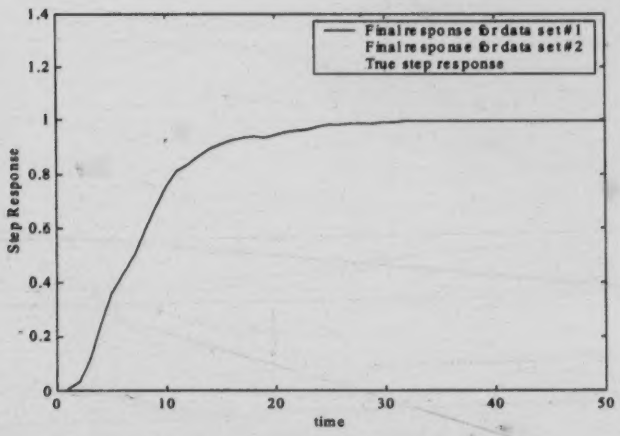


Figure 6.5: Models from the iterations

6.5 Conclusions and Future work

In this chapter we have discussed in detail various properties of the control relevant predictor obtained by minimizing a multi-step ahead prediction error objective function. A new method for iterative identification and control in the MPC framework is proposed and a few properties of this algorithm are proved. There are a number of issues that need to be addressed especially in the case of iterative identification and control. The effect of tuning noise model on the closed loop performance and on the process model needs to be studied in detail. The cross term in (6.22) explains the amount of correlation between the control and identification errors. In order to explain the improvement of performance during the iterations a thorough understanding of the cross term in both the time and frequency domains is necessary. A better understanding of the convergence of the iterations is also useful. Since the optimization problem is highly nonlinear, it is very likely that we get stuck at a local minimum. Therefore, the initial guess of the process parameters has an impact on the final closed loop performance achieved. This issue needs further research.

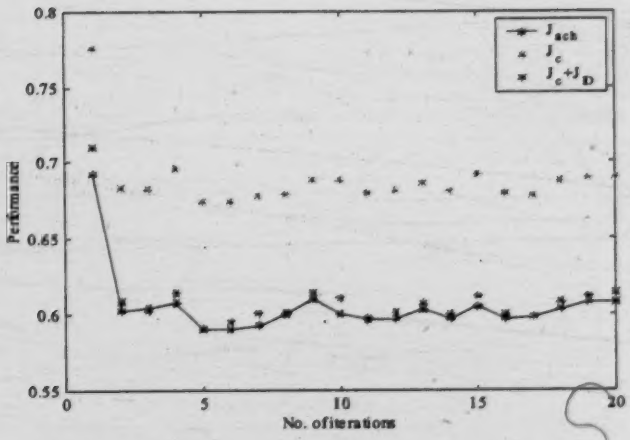


Figure 6.6: Control and Identification performances vs No. of iterations

7

An Approximation of the Adaptive Dual Control Problem and Interpretations

This chapter presents an approximation of the adaptive dual control problem for general prediction error models. A number of interpretations are presented. We show that the approximate dual control objective function is equivalent to choosing a special weighting matrix for the inputs in the traditional model predictive control objective function. [1]

7.1 Introduction - A historical perspective

Over the last 40 years there has been a great development in the research on adaptive control theory and its applications. Many of the developed adaptive algorithms have been successfully applied in various engineering fields. However, most of these adaptive control strategies are based on certainty equivalence principle *i.e.*, a model based controller is designed under the assumption that the identified model is accurate and there is no uncertainty in the estimated model. The estimated model parameters are used in the control algorithm as if they are the "true" parameters. The certainty equivalence principle is easy to implement and therefore has been applied widely. Nevertheless, it can lead to poor performance especially when the model uncertainty is large.

[1] A part of this chapter was presented at CSChE 2001, Halifax (Gopaluni *et al.* 2001*a*).

In the early 60s, Feldbuam (Feldbaum 1960-1961) posed an optimal adaptive control problem that takes the model uncertainty into account. He showed that certainty equivalence approach is not always optimal and in fact can lead to unacceptable performance. He postulated two main properties that an optimal adaptive controller should have

- The controller must be *cautious* in tracking the set point
- At the same time, the controller must excite the process in an optimal way to improve the model accuracy for better performance in the future

Adaptive control systems with the above features are named adaptive dual control systems.

An exact solution to the dual control problem has eluded researchers for the last four decades. The original problem requires a dynamic programming solution but the problem is too complex to solve either analytically or numerically due to growing dimension of the underlying space (sometimes called the curse of dimensionality). Exact solutions for a simple system with a few possible states are obtained by Sternby (1976). The intractability of this problem for higher dimensional systems has lead to two practical approaches to solve this problem

- those based on various approximations of the optimal dual control problem
- and those based on easy to solve reformulation of the original problem with its dual properties in tact

These approaches are named implicit and explicit adaptive dual control methods respectively. The main idea of these methods is to provide a control algorithm that has the dual properties even though it is not optimal.

The are a number of approximations proposed to solve the dual control problem (Bar-Shalom and Tse 1974, Bayard and Eslami 1985, Bertsekas 1976, Birmiwal 1994, Unbehauen 2000) but most of them lose the dual features of the original problem and the control performance remains inadequate. On the other hand, reformulated adaptive control methods use a modified objective function with two terms in it - one that accounts for the control error and the other for the model uncertainty.

7.2 Preliminaries

Many of the commonly used adaptive control algorithms assume that the identified model is known exactly (*i.e.*, they implicitly invoke the certainty equivalence principle). Even if there is an update law for the parameters, they try to minimize the difference between

the estimated and the true parameters. However, the update laws do not consider the effect of the parameter variance on the performance. This can lead to poor performance. The 'smaller' the covariance matrix the better is the performance. In many adaptive methods, inputs are designed without regard for their effect on the parameter variance thus leading to poor performance. The inputs, if designed in such a way that they try to reduce the parameter variance and at the same time achieve control objectives (Feldbaum 1960-1961), can result in better performance in the long term. This problem is known as the dual control problem.

Intuitively, the best control signal is one which perturbs the plant to gain more information about the process and at the same time achieves the control objectives. But the two goals are conflicting, a knowledge of process parameters can only be obtained by proper excitation where as control requires little excitation. This trade-off between the input signal design for estimation and control is the central idea of dual control.

The exact solution to this problem is given by dynamic programming which is hard to solve either analytically or numerically. An approximation which results in a semi-definite program for a static MISO model was introduced in Lobo and Boyd (1999). A generalization of this idea to deal with dynamic models with stochastic inputs will be presented and it will be shown that the same results can be obtained by a much simpler method. Also, a number of interesting interpretations of this new approximation in the context of cautious control will be presented.

7.2.1 Feldbaum's Optimal Dual Control Problem

Feldbaum's optimal dual controller aims to control an uncertain stochastic system described by the following discrete-time state space model

$$\begin{aligned} x(t+1) &= f_t(x(t), u(t), w(t), \theta(t)) \\ y(t) &= h_t(x(t), v(t)) \end{aligned} \tag{7.1}$$

where $x(t) \in \mathrm{R}^n$ are the states, $y(t) \in \mathrm{R}^m$ are the outputs and $u(t) \in \mathrm{R}^p$ are the inputs. $\theta(t)$ is the vector of varying process parameters. It is generally assumed that the process states and the measured outputs are corrupted by white noise sequences $w(t)$ and $v(t)$ respectively with variances, σ_w^2 and σ_v^2. f_t and h_t are two time varying vector functions of the process states and noise.

At any time instant, t, it is assumed that the past process inputs and outputs are known exactly and the $\sigma-$algebra generated by the past input/output sequence

is denoted by

$$\mathcal{F}_t = \{y(t), y(t-1), \cdots, y(0), u(t-1), u(t-2), \cdots, u(0)\} \tag{7.2}$$

The performance index is defined as

$$J = E\left[\sum_{t=0}^{N-1} s_{t+1}(x(t+1), u(t))\right] \tag{7.3}$$

where $s_{k+1}[.,.]$ are known positive scalar functions. The expectation is taken with respect to all the random signals in the process and N is the horizon over which the performance is optimized.

Now the problem of optimal adaptive dual control consists of finding the control policy $u(k) = u(k, \mathcal{F}_k)$ for $k = 0, 1, \cdots, N-1$ that minimizes the above performance index for the system described in (7.1). The optimal solution of the above problem can be obtained by solving the following recursive stochastic dynamic programming problem.

$$J_{t-1}(\mathcal{F}_{N-1}) = \min_{u(N-1)} E\left[s_N(x(N), u(N-1))|\mathcal{F}_{N-1}\right] \tag{7.4}$$

$$J_t(\mathcal{F}_t) = \min_{u(t)} E\left[s_{t+1}(x(t+1), u(t)) + J_{t+1}(\mathcal{F}_{k+1})|\mathcal{F}_k\right] \tag{7.5}$$

$$\text{for } t = N-2, N-3, \cdots, 0 \tag{7.6}$$

The above problem is difficult to solve analytically even for low dimensional problems. The numerical methods also become intractable as the size of N increases due to the many fold increase in the dimensionality of the underlying space even for simple cases (Bar-Shalom and Eslami 1985, Bayard *et al.* 1992). However, a thorough analysis of this approach enables us to find the inherent dual properties of the control signal which in turn can be used to design approximate dual controllers. A simple example to demonstrate the dual properties of the above control problem is presented below.

Example 7.2.1 Consider a SISO system described by the following discrete model

$$y(t+1) = \theta u(t) + e(t) \tag{7.7}$$

where $e(t)$ is a white noise sequence and $\theta \neq 0$. θ is the unknown parameter with the initial estimate $\hat{\theta(0)}$ and the initial covariance estimate $\Sigma(0)$. The following performance objective function needs to be minimized

$$J = E\left[\sum_{t=1}^{N} (r(t) - y(t))^2\right] \tag{7.8}$$

where $r(t)$ denotes the known setpoint trajectory. The resulting optimal control problems is unsolvable for large N. The optimal parameter estimate can however be obtained using the Kalman filter in the following form (Ljung 1999)

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \frac{\Sigma(t)u(t)}{\Sigma(t)u^2(t)+\sigma_e^2}[y(t+1) - \hat{\theta}(t)u(t)] \tag{7.9}$$

$$P(t+1) = \frac{\Sigma(t)\sigma_e^2}{\Sigma(t)u^2(t)+\sigma_e^2} \tag{7.10}$$

It is easy to observe (c.f (7.9) and (7.10)) that, for fixed σ_e^2, large values of input improve the estimation and for an unbounded input signal, the exact estimate of the parameter can be obtained after only one measurement. It can also be observed that large values of input reduce the parameter variance.

$$\lim_{|u(t)|\to\infty} P(t+1) = 0 \tag{7.11}$$

$$\lim_{|u(t)|\to\infty} \hat{\theta}(t+1) = \theta \tag{7.12}$$

Therefore, a large value of the input improves the quality of the estimates. The dual control problem is to excite the process optimally to obtain enhanced performance.

If we were to use certainty equivalence approach, it is assumed that all stochastic variables in the system are equal to their expectations. Thus, with the certainty equivalent assumption the optimal control law has the form

$$u(t) = \frac{r(t+1)}{\hat{\theta}(t)} := u_{CE}(t) \tag{7.13}$$

On the other hand, the minimization of one-step cost function (*i.e.*, $N = 1$) would result in an optimal law given by

$$u(t) = \frac{1}{1+\Sigma(t)/\hat{\theta}(t)^2} u_{CE}(t) \tag{7.14}$$

This input is always smaller in magnitude than the input generated by an equivalent certainty equivalence controller. The larger the uncertainty in the estimated parameters, the smaller the magnitude of the input generated by this controller. Thus, giving it the name - *cautious controller*. Thus, controllers with the two properties - cautiousness and persistent excitation are called adaptive dual controllers.

A typical dual control system and a typical adaptive control system are shown in figs.7.1 and 7.2

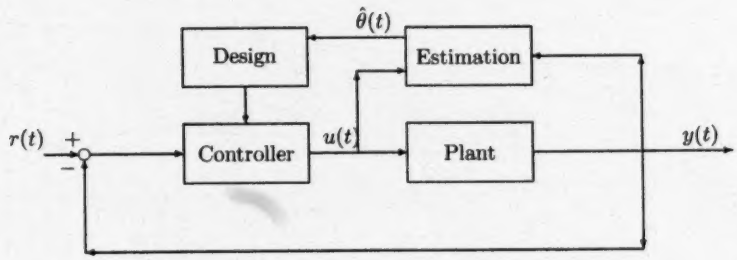


Figure 7.1: An adaptive control system

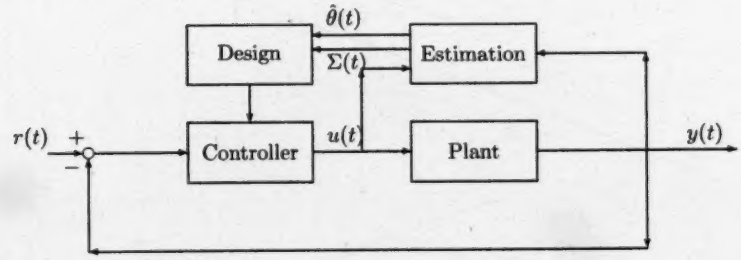


Figure 7.2: An adaptive dual control system

7.3 An approximation of Dual control problem for FIR models

7.3.1 Problem Formulation

Consider the following FIR model

$$y(k) \;:=\; h_f^T u_f^{(k-1)} + h_p^T u_p^{(k-1)} + e(k) \tag{7.15}$$

where 'f' stands for future and 'p' stands for past. $u_f^{(k)}$ consists of all the unknown inputs at time instant k and h_f is a vector of the corresponding impulse response coefficients. h_p and u_p are similarly defined. For instance, for a SISO system, $u_f^{(k)} = u(k)$ and $u_p^{(k)} = [u(k-1)\cdots u(k-n+1)]^T$, where n is the model order. Note that this definition takes into account the multi input nature of the system. e_k represents random gaussian white noise with zero mean and variance σ_e^2. The system parameter vector $h^T = [h_f \quad h_p]^T$ is assumed to be unknown with *a priori* distribution $\mathcal{N}(\hat{h}_0, \Sigma_0)$, where

$\mathcal{N}$ stands for normal distribution and $\hat{h}_0$ represents the initial estimate of the impulse response coefficients. We will also assume that the covariance matrix is positive definite and the inverse of the covariance matrix, popularly known as *information matrix* is represented as

$$\Pi_0 = \Sigma_0^{-1} \tag{7.16}$$

For the sake of simplicity we assume that the model is MISO *i.e.*, $u(k) \in \mathbb{R}^m$, m being the number of inputs. The objective is to optimize a performance measure which is a function of the true outputs, $y(1), y(2), \cdots, y(P)$ where P denotes the horizon over which the performance measure needs to be optimized. In this chapter, we are interested in minimizing the expected value of the sum of the squares of the control errors and the squared inputs *i.e.*,

$$J^{(P)} = E\left[\sum_{t=1}^{P}(r(t) - y(t))^2 + \lambda u(t)^T u(t)\right] \tag{7.17}$$

where the expectation is taken over the distributions of h^T and the noise. Let $\mathcal{F}_t$ be the $\sigma-$algebra generated by all the past inputs and outputs and past/future setpoint excitation *i.e.*, $\mathcal{F}_t = \{y(0), y(1), \cdots, y(t), u(0), \cdots, u(t-1), r(0), \cdots, r(P)\}$. A feasible input is one which is a function of $\mathcal{F}_t$ *i.e.*, $u(k) := u(k, \mathcal{F}_k)$.

Intuitively, the best way to optimize the above objective function is to find an input that drives the output as close as possible to the setpoint. Since the model parameters are not known precisely, some amount of perturbation is also needed to improve the quality of the process parameters at the next instant for control. These are two conflicting strategies. For instance, if the setpoint is zero the desired output is obtained by holding the input at zero. However, it is the least informative input. Therefore, an input that can provide better excitation for better estimation is useful in obtaining good tracking performance in the future. The philosophy here is to sacrifice short term performance for long term gains. Obtaining an input that optimally accounts for both estimation and good control is the concern of this chapter.

The true solution to the above problem can be obtained by numerically solving a dynamic programming problem. Unfortunately, as the size of P increases, the problem becomes intractable (called "the curse of dimensionality" in the optimization parlance). In this chapter, we extend the approximation presented by Lobo and Boyd (1999) to various types of dynamic models by presenting a new interpretation of their approximation. We also show that this method can be employed in developing an adaptive Model Predictive Controller.

The following update laws are used for the parameters and their covariance (Ljung 1999)

$$\Pi_{t+1} = \Pi_0 + \sigma_e^{-2} U(t)^T U(t) \qquad \hat{h}_{t+1} = \Pi_{t+1}^{-1}(\Pi_t \hat{h}_t + \sigma_e^{-2} U(t)^T Y(t)) \quad (7.18)$$

where

$$U(t) := \begin{bmatrix} u(1)^T \\ u(2)^T \\ \vdots \\ u(t)^T \end{bmatrix} \qquad Y(t) := \begin{bmatrix} y(1)^T \\ y(2)^T \\ \vdots \\ y(t)^T \end{bmatrix} \quad (7.19)$$

The above update laws, can equivalently be written as the following recursive formulas

$$\Pi_{t+1} = \Pi_t + \sigma_e^{-2} u(t+1) u^T(t+1) \quad (7.20)$$

$$\hat{h}_{t+1} = \hat{h} + \sigma_e^{-2} \Pi^{-1} u(t+1)(y(t+1) - \hat{h}_t u(t+1)) \quad (7.21)$$

Effect of parameter variance on closed loop performance

In this section we will show that the parametric covariance has a rather direct impact on the expected value of the following objective function.

$$J^{(P)} := E\left(\sum_{t=1}^{P} (y(t) - r(t))^2 + \sum_{t=1}^{P} \lambda u(t)^T u(t)\right) \quad (7.22)$$

Using the model in (7.15), we get

$$J^{(P)} = E\left(\sum_{t=1}^{P} \lambda u(t)^T u(t) + \sum_{t=1}^{P} (\tilde{h}^T \underline{u}_t + \hat{h}^T \underline{u}_t - r(t))^2\right) + P\sigma_e^2 \quad (7.23)$$

where $\tilde{h} = h - \hat{h}$ and $\underline{u}_t = [u(t-1) \cdots u(t-n)]$. σ_e^2 is the error variance. Assuming that $h \sim \mathcal{N}(\hat{h}, \Sigma)$

$$J^{(P)} = \sum_{t=1}^{P} \lambda u(t)^T u(t) + \sum_{t=1}^{P} \underline{u}_t^T \Sigma \underline{u}_t + (\hat{h}^T \underline{u}_t - r(t))^2 + P\sigma_e^2 \quad (7.24)$$

Clearly from the above expression the expected value of the cost function is directly proportional to the covariance matrix. The larger the covariance is, the higher is the value of the cost function. If we were to use certainty equivalence principle we would assume that the parameter variance is zero and as a result the true cost function can be much larger than the assumed cost function depending on the parameter variance.

Before introducing the approximation of the dual control problem and its interpretations, let us look at a few non-optimal or heuristic ways of introducing cautiousness into the control algorithm. Instead of totally ignoring the input it is possible to regularize the input by minimizing only the immediate expected cost given the conditional distribution of the parameters (Lobo and Boyd 1999) *i.e.*,

$$\begin{aligned} E\left[(y(k)-r(k))^2+\lambda u(k)^T u(k)|\mathcal{F}_k\right] &= u(k)^T\Pi_{k-1}^{-1}u(k)+[\hat{h}_{k-1}u(k)-r(k)]^2+\sigma_e^2 \\ &\quad +\lambda u(k)^T u(k) \end{aligned} \tag{7.25}$$

The minimizing input, obtained by differentiating the above objective function and equating to zero, is

$$u(k) = [\Pi_{k-1}^{-1}+\hat{h}_{k-1}\hat{h}_{k-1}^T+\lambda I]^{-1}\hat{h}_{k-1}r(k) \tag{7.26}$$

The term Π^{-1} can be seen as a regularization term. For large parameter variance, this term ensures that the controller is not aggressive (cautious control) and for small variance this input approaches the certainty equivalent input. Similar sub-optimal and heuristic controllers can be designed by adding a weighted information matrix to the objective function or by using dithering signals.

In the following paragraphs we outline the derivation of the dynamic programming solution involved in solving the dual control problem. For the sake of clarity in explanation we assume that $\lambda = 0$.

7.3.2 Approximation

Optimal solution for P = 1:

$$\begin{aligned} J^{(1)} &:= E\left([y(1)-r(1)]^2\right) \\ &= u_f^{(0)^T}\Sigma_f^{(0)}u_f^{(0)}+u_p^{(0)^T}\Sigma_p^{(0)}u_p^{(0)}+2u_f^{(0)^T}\Sigma_{fp}^{(0)}u_p^{(0)}+(\hat{h}_f^{(0)^T}u_f^{(0)}+\hat{h}_p^{(0)^T}u_p^{(0)}-r(1))^2+\sigma_e^2 \end{aligned} \tag{7.27}$$

The minimizing input can easily be obtained by differentiating the above objective function and equating it zero.

$$u_f^{(0)} = (\Sigma_f^{(0)}+\hat{h}_f^{(0)}\hat{h}_f^{(0)^T})^{-1}\left[\hat{h}_f^{(0)}r(1)-\hat{h}_f^{(0)}\hat{h}_p^{(0)^T}u_p^{(0)}-\Sigma_{fp}^{(0)}u_p^{(0)}\right] \tag{7.28}$$

where

$$\Pi_k^{-1} = \begin{bmatrix} \Sigma_f^{(k)} & \Sigma_{fp}^{(k)} \\ \Sigma_{pf}^{(k)} & \sigma_p^{(k)} \end{bmatrix} = \Sigma_k \tag{7.29}$$

Substituting the optimal input into the original objective function in (7.27), we get

$$J_*^{(1)} := \sigma_e^2 + \frac{r(1)^2 + u_p^{(0)^T}\Sigma_p u_p^{(0)} - u_p^{(0)^T}\Sigma_{fp}\Sigma_f^{-1}\Sigma_{pf}u_p^{(0)} + \hat{h}^{(0)^T}Q^{(0)} + \hat{h}^{(0)^T}P^{(0)}\hat{h}^{(0)}}{1 + \hat{h}_f^{(0)^T}\Sigma_f^{-1}\hat{h}_f^{(0)}} \tag{7.30}$$

Full details regarding the derivation and the notation for some of the terms used in the above expressions is presented in the appendix.

Optimal solution for P = 2:

$$\begin{aligned} J^{(2)} &= E\left((y(1) - r(1))^2 + (y(2) - r(2))^2\right) \\ &= u_f^{(0)^T}\Sigma_f^{(0)}u_f^{(0)} + u_p^{(0)^T}\Sigma_p^{(0)}u_p^{(0)} + 2u_f^{(0)^T}\Sigma_{fp}^{(0)}u_p^{(0)} + (\hat{h}_f^{(0)^T}u_f^{(0)} + \hat{h}_p^{(0)^T}u_p^{(0)} - r(1))^2 \\ &\quad + E\left(u_f^{(1)^T}\Sigma_f^{(1)}u_f^{(1)} + u_p^{(1)^T}\Sigma_p^{(1)}u_p^{(1)} + 2u_f^{(1)^T}\Sigma_{fp}^{(1)}u_p^{(1)} + (\hat{h}_f^{(1)^T}u_f^{(1)} + \hat{h}_f^{(1)^T}u_p^{(1)}\right. \\ &\quad \left. - r(2))^2 \,|\mathcal{F}_1\right) \end{aligned} \tag{7.31}$$

Therefore the dynamic programming problem for $P = 2$ is

$$\begin{aligned} \min_{u_f^{(0)}} J^{(2)} &= \min_{u_f^{(0)}}\left(u_f^{(0)^T}\Sigma_f^{(0)}u_f^{(0)} + u_p^{(0)^T}\Sigma_p^{(0)}u_p^{(0)} + 2u_f^{(0)^T}\Sigma_{fp}^{(0)}u_p^{(0)} + (\hat{h}_f^{(0)^T}u_f^{(0)}\right. \\ &\quad \left. + \hat{h}_p^{(0)^T}u_p^{(0)} - r(1))^2 + E\left[\min_{u_f^{(1)}} J^{(1)}(\hat{h}^{(1)}, \Pi_1, \sigma_e^2, r(2)) \,|\mathcal{F}_1\right]\right) \end{aligned} \tag{7.32}$$

The inner minimization involves an integral of the form

$$E_X\left[\frac{a_0X^2 + a_1X + a_2}{b_0X^2 + b_1X + b_2}\right] \tag{7.33}$$

where a_i's and b_i's are constants and $X \sim \mathcal{N}(0, \sigma_X^2)$. This integral can be calculated numerically, but if P is large it may get really complicated. Therefore an approximation of the form

$$\left[\frac{a_0X^2 + a_1X + a_2}{b_0X^2 + b_1X + b_2}\right] \approx \frac{a_2}{b_2} \tag{7.34}$$

is introduced. This approximation reduces the objective function to

$$\begin{aligned} \min_{u_f^{(0)}} \tilde{J}^{(2)} &:= \min_{u_f^{(0)}}\left(u_f^{(0)^T}\Sigma_f^{(0)}u_f^{(0)} + u_p^{(0)^T}\Sigma_p^{(0)}u_p^{(0)} + 2u_f^{(0)^T}\Sigma_{fp}^{(0)}u_p^{(0)} + (\hat{h}_f^{(0)^T}u_f^{(0)}\right. \\ &\quad \left. + \hat{h}_p^{(0)^T}u_p^{(0)} - r(1))^2 + J_*^{(1)}(\hat{h}^{(1)}, \Pi_1, \sigma_e^2, r(2))\right) \end{aligned} \tag{7.35}$$

where $J_*^{(1)}(\hat{h}^{(1)}, \Pi_1, \sigma_e^2, r(2))$ is obtained by replacing $\hat{h}^{(0)}$ by $\hat{h}^{(1)}$, Π_0 by Π_1 and $r(1)$ by $r(2)$ in (7.30). Now "undoing" the minimization over $u_f^{(1)}$ in (7.35), we see that (7.35) is equivalent to

$$\begin{aligned}\min_{u_f^{(0)}, u_f^{(1)}} \tilde{J}^{(2)} &:= \min_{u_f^{(0)}, u_f^{(1)}} \Big(u_f^{(0)^T} \Sigma_f^{(0)} u_f^{(0)} + u_p^{(0)^T} \Sigma_p^{(0)} u_p^{(0)} + 2 u_f^{(0)^T} \Sigma_{fp}^{(0)} u_p^{(0)} + (\hat{h}_f^{(0)^T} u_f^{(0)} + \\ &\quad \hat{h}_p^{(0)^T} u_p^{(0)} - r(1))^2 + u_f^{(1)} \Sigma_f^{(1)} u_f^{(1)} + u_p^{(1)^T} \Sigma_p^{(1)} u_p^{(1)} + 2 u_f^{(1)^T} \Sigma_{fp}^{(1)} u_p^{(1)} + \\ &\quad (\hat{h}_f^{(1)^T} u_f^{(1)} + \hat{h}_p^{(1)^T} u_p^{(1)} - r(2))^2 \Big) \qquad (7.36) \\ &= \min_{u_f^{(0)}, u_f^{(1)}} \Big(u^{(0)^T} \Sigma_0 u^{(0)} + u^{(1)^T} \Sigma_1 u^{(1)} + (\hat{h}_0^T u^{(0)} - r(1))^2 + (\hat{h}_1^T u^{(1)} - r(2))^2 \Big) \qquad (7.37)\end{aligned}$$

where $u^{(0)} = [u_f^{(0)} \quad u_p^{(0)}]$. According to the covariance update law (7.20), the covariance matrix is independent of the white noise entering the process. Therefore, comparing (7.31) with (7.36), we see that the proposed approximation is equivalent to the assumption that $\hat{h}_k = \hat{h}_0$ but on the other hand the covariance matrix changes with the input. In other words, we are assuming that the effect of mean changes in the parameter vector are negligible compared to the effect of parameter variance on the performance objective function.

For $P > 2$, one has to include more terms of the above form, however the resulting quadratic equation is not convex. Therefore an approximation of that equation around some nominal input can help us in converting this problem to a semi-definite programming problem with input/output constraints. For $P > 2$ the approximate objective function becomes

$$\begin{aligned}\min_{u_f^{(0)}, \cdots, u_f^{(P-1)}} \tilde{J}^{(P)} &:= \min_{u_f^{(0)}, \cdots, u_f^{(P-1)}} \Big(u^{(0)^T} \Sigma_0 u^{(0)} + \cdots + u^{(P-1)^T} \Sigma_{P-1} u^{(P-1)} + (\hat{h}_0^T u^{(0)} - r(1))^2 \\ &\quad + \cdots + (\hat{h}_{P-1}^T u^{(P-1)} - r(P-1))^2 \Big) \qquad (7.38)\end{aligned}$$

Example 7.3.1 The above approximation is tested on the following model with two inputs, one output and two impulse response coefficients.

$$y(k) = h^T [u_1(k-1) \;\; u_2(k-1) \;\; u_1(k-2) \;\; u_2(k-2)]^T \qquad (7.39)$$

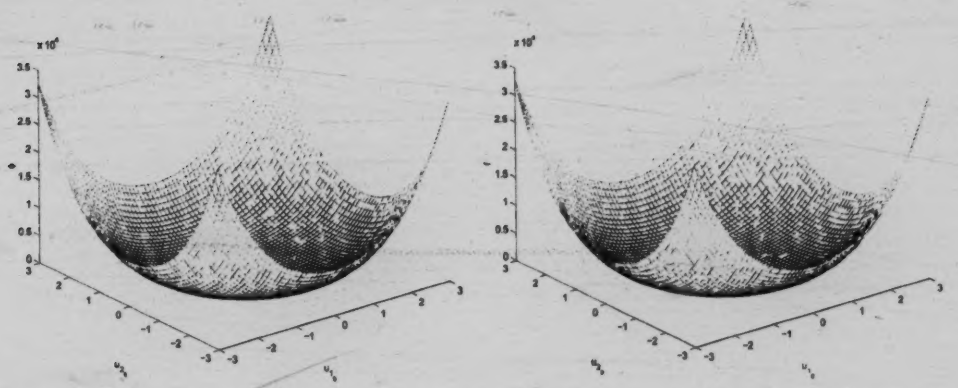
with

$$\Pi_0 = \begin{bmatrix} 6 & 2 & 1 & 3 \\ 2 & 10 & 1 & 1.5 \\ 1 & 1 & 8 & 2 \\ 3 & 1.5 & 2 & 15 \end{bmatrix} \qquad \hat{h}_0 = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.1 \\ 0.3 \end{bmatrix} \qquad (7.40)$$

and

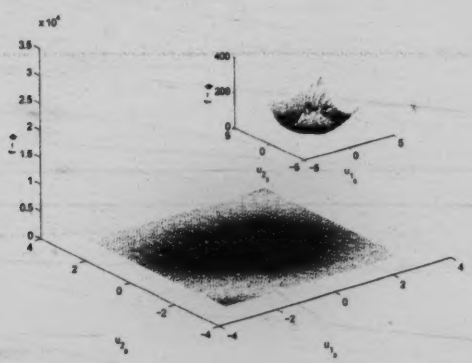
$$r = [0 \;\; 10] \qquad\qquad \sigma_e^2 = 0.01$$

The expected cost as a function of the two inputs is plotted in fig.7.3. The approximation seems to be very good. In fact, as noted in Lobo and Boyd (1999), the approximation error is minimum around the optimal input (see fig.7.3(c)).



(a) Exact expected cost as a function of the two inputs

(b) Approximate expected cost as a function of the two inputs



(c) Difference between the approximate and the true solutions - with different scales

Figure 7.3: The true and the approximate expected costs as a function of inputs

7.4 Interpretations of the approximation

In the previous section we have proposed an approximation of the dual control objective function. This particular approximation can be interpreted in a number of ways. Using these interpretations it is possible to extend the above approximation to models other than FIR models. The crucial part of the approximation in the previous section was

$$E_X\left[\frac{a_0X^2+a_1X+a_2}{b_0X^2+b_1X+b_2}\right] \approx \frac{a_2}{b_2} \tag{7.41}$$

Note that $X \sim \mathcal{N}(0,\sigma_X^2)$. Using the Taylor's series expansion around $X=0$, we get

$$\begin{aligned}\frac{a_0X^2+a_1X+a_2}{b_0X^2+b_1X+b_2} &= \frac{a_2}{b_2}+\left(\frac{a_1a_2}{b_2}-\frac{a_2b_1}{b_2^2}\right)X+\cdots+\cdots \\ &:= \tilde{c}_0+\tilde{c}_1X+\tilde{c}_2X^2+\cdots \end{aligned} \tag{7.42}$$

Therefore

$$E_X\left[\frac{a_0X^2+a_1X+a_2}{b_0X^2+b_1X+b_2}\right] = \tilde{c}_0+\tilde{c}_1E[X]+\tilde{c}_2E[X^2]+\cdots \tag{7.43}$$

Theorem 7.4.1 (Casella and Berger (1990)) *Let $X \sim \mathcal{N}(\theta,\sigma^2)$ and let g be a differentiable function satisfying $E[g'(X)] < \infty$. Then*

$$E[g(X)(X-\theta)] = \sigma^2 Eg'(X) \tag{7.44}$$

Therefore

$$E[X^n] = \begin{cases} 1.3.5.\cdots.(n-1)\sigma^n & \text{for even } n \\ 0 & \text{for odd } n \end{cases} \tag{7.45}$$

which implies

$$E_X\left[\frac{a_0X^2+a_1X+a_2}{b_0X^2+b_1X+b_2}\right] = \tilde{c}_0+\tilde{c}_2\sigma_X^2+\tilde{c}_4\sigma_X^4+\cdots \tag{7.46}$$

In order to make the original dual control problem amenable to available optimization techniques, we have approximated the above integral with $\tilde{c}_0$. Clearly, the above approximation can be poor if the variance of the random variable, X, is large. It is shown in the appendix that this random variable is a function of the process noise and therefore its variance depends on the process noise variance.

As mentioned earlier this approximation is equivalent to the assumption $\hat{h}_0 = \hat{h}_1 = \cdots = \hat{h}_{P-1}$. Under this assumption the approximate objective function can be written as

$$\begin{aligned}\tilde{J}^{(P)} &= E\left[(y(1) - r(1))^2\right] + E\left[(y(2) - r(2))^2 | y(1) = \hat{y}(1)\right] \\ &\quad + E\left[(y(3) - r(3))^2 | y(1) = \hat{y}(1), y(2) = \hat{y}(2)\right] + \cdots\end{aligned} \tag{7.47}$$

The above version of the approximation follows from the fact that $y(1) = \hat{y}(1)$,$y(2) = \hat{y}(2)$,$\cdots$, implies

$$\begin{aligned}\hat{h}_1 &= \Pi_1^{-1}(\Pi_0 \hat{h}_0 + \sigma_e^{-2} u(1) y(1)) \\ &= \Pi_1^{-1}(\Pi_0 \hat{h}_0 + \sigma_e^{-2} u(1) u(1)^T \hat{h}_0) \quad [\text{if } y_1 = \hat{y}_1] \\ &= \Pi_1^{-1} \Pi_1 \hat{h}_0 \\ &= \hat{h}_0\end{aligned} \tag{7.48}$$

and through induction, it is possible to show that $\hat{h}_0 = \hat{h}_1 = \hat{h}_2 = \cdots = \hat{h}_k = \cdots$. This interpretation of the original approximation can be used to extend it to different types of process models *viz.*, ARX, BJ.

7.4.1 Certainty equivalence objective function and the approximation

It is interesting to see how the certainty equivalence objective function compares with the approximation of the dual control objective function. For instance, if $P = 2$, the certainty equivalence objective function is

$$\min_{u_f^{(0)}, u_f^{(1)}} J = (\hat{h}_0^T u^{(0)} - r(1))^2 + (\hat{h}_0^T u^{(1)} - r(2))^2 \tag{7.49}$$

and the approximate dual control objective function is

$$\min_{u_f^{(0)}, u_f^{(1)}} \tilde{J}^{(2)} = (\hat{h}_0^T u^{(0)} - r(1))^2 + (\hat{h}_1^T u^{(1)} - r(2))^2 + u^{(0)^T} \Pi_0^{-1} u^{(0)} + u^{(1)^T} \Pi_1^{-1} u^{(1)} \tag{7.50}$$

Observe that the terms $u^{(0)^T} \Pi_0^{-1} u^{(0)}$ and $u^{(1)^T} \Pi_1^{-1} u^{(1)}$ in the approximate dual control objective function are nothing but the "u^2" terms in the traditional MPC objective function with a special weighting which is the inverse of information matrix (or covariance matrix). This input weight is large if the covariance matrix is "large" in

some sense. The controller tries to be cautious if the parameters are not known with high precision (*i.e.*, covariance is large), thus introducing the cautiousness of the dual control algorithm into the control objective function.

7.5 Conclusions

We have extended the approximation of Lobo and Boyd (1999) to general prediction error models and provided number of interpretations. We have also shown that the approximation is equivalent to using the covariance matrix as the weight on inputs in the objective function. The approximate objective function is not convex in the inputs and hence it is not easy to optimize it with respect to the input. First order approximation with respect to the covariance matrix leads to a linear matrix inequality. It is then possible to use this approximation to develop an algorithm for adaptive MPC. This forms part of the future work.

8

Model Free Iterative Controller Tuning

A new method for model free controller tuning is proposed in this chapter. The proposed method is an improvement of the popular Iterative Feedback Tuning (IFT) by Hjalmarsson *et al.* (1998). Original IFT algorithm requires an estimate of the derivative of the closed loop performance objective function and is applicable only to linear systems with a particular performance objective function. The method presented in this chapter, is derivative free and as a result works for any given objective function. A few illustrative examples supporting the arguments in this chapter are also presented.

8.1 Introduction

In the first few chapters, an algorithm to iteratively improve the closed loop performance by model and controller tuning was developed for model predictive controllers. This method involves two steps - identification of a new model and redesigning a controller. An optimal controller that achieves these objectives invariably depends on the process model and the disturbance characteristics. Such an optimal controller, therefore, may be high order depending on the order of the process model and the noise model. Instead, it is possible to directly identify a controller that meets the closed loop performance objectives.

Most of the control objectives can be expressed in terms of an objective function.

Generally, due to the stochastic nature of this objective function and due to lack of reliable models it is not possible to obtain an explicit optimal solution without making use of certainty equivalence principle. Even if an explicit solution is obtained, the optimal solution may not have the desired controller structure.

There has been a lot of work on obtaining desired control structure and order using various controller order reduction techniques (Anderson and Liu 1989). However, most of these methods depend on the availability of process and noise models. The book by Astrom and Hagglund (1995) also provides some methods for tuning PID controllers by imposing constraints derived from performance objectives on the closed loop nyquist plots.

More recently, a new method called Iterative Feedback Tuning (IFT) was proposed by Hjalmarsson *et al.* (1998) to design reduced complexity controllers without identifying the process model. This approach is very similar to direct adaptive control (Sastry and Bodson 1989) the only difference being complexity of the controller obtained. In the IFT method, the emphasis is on obtaining controllers of pre-specified structures. On the other hand, in direct adaptive control the designed controller structure depends on the actual process and noise model orders. The idea behind IFT is to estimate the derivative of the closed loop performance objective function. This estimate is then used in a Gauss-Newton method for optimization to obtain model parameter updates.

There are a number of advantages of using this method. This method can be directly used to identify controller parameters (for example for PID controllers). The Gauss-Newton method used in IFT is guaranteed to converge due to the unbiased estimate of the gradient of the performance objective function. However, if the signal to noise ratio is small, the estimated derivatives are prone to large variance errors leading to slower convergence. Especially for integrating controllers, these errors can be large as the gradient estimates are obtained by filtering functions of input - output data through the inverse of the controller. It is important to note that even for small signal to noise ratios, it is possible to reduce the variance in the objective function by increasing the duration of closed loop experiments. However, our aim is to obtain the best possible closed loop performance through minimum interference to the normal process operating conditions. For processes, with minor nonlinearities (for instance in the valves), the gradient estimates may not be unbiased, which is an essential property required to ensure convergence. Moreover, in order to estimate this gradient, IFT conducts three experiments on the closed loop system. Therefore, if the algorithm converges in n iterations, the total number of experiments conducted is $3n$. Since experiments are expensive, it is generally desirable to keep the number of experiments to a minimum.

In light of these disadvantages associated with the current IFT algorithm, a new algorithm based on Nelder-Meade direct search optimization method for directly identifying the controller by optimizing the closed loop performance is proposed in this chapter. Since this method does not use gradients, some of the disadvantages associated with the IFT algorithm can be avoided. It has been observed that for the same number of experiments the proposed method provides better performing controllers for processes with small signal to noise ratio. Moreover, unlike IFT, this method works even if the process has mild nonlinearities since no derivatives are estimated.

8.2 Problem formulation

Consider the closed loop system shown in fig.8.1 where the process is allowed to be any nonlinear function of inputs, $G(u)$. $H(e(t))$ is some function of gaussian distributed white noise, $e(t)$. In general, most of the control objectives can be expressed as some function of the process and noise models. Denote the control objective function by $J(C, G, H, e(t), r(t))$. The dependence of this objective function on process and noise models, noise, set point excitation and the controller is shown explicitly. Since it is a function of the noise, it is stochastic in nature. In any certainty equivalence formulation an estimate of process and noise models is obtained to optimize a deterministic equivalent of the above objective function. In this chapter, optimization of the original stochastic objective function is of concern.

For any given controller it is possible to get a stochastic realization of the performance objective function, J, by conducting experiments on the process under closed loop. Each closed loop experiment provides a single realization of the stochastic objective function. The idea behind iterative feedback tuning and other direct control tuning methods is to conduct as many closed loop experiments as required to evaluate J for various controllers and use those values to estimate optimal controller tunings.

The problem of direct controller tuning can now be posed as an unconstrained stochastic optimization problem

$$\min_{C} E\left[J(C, G, H, e(t), r(t))\right] \tag{8.1}$$

where there is no explicit expression for the objective function, $E[J]$, as a function of the process model and other variables. Nevertheless, it is possible to obtain a stochastic realization of the objective function through experiments. Our objective is to find the optimal controller, $C(q)$, by conducting as few experiments as possible.

There are a number of stochastic optimization algorithms available for solving

problems of this type. In Hjalmarsson *et al.* (1998) they use an algorithm similar to Robbins-Monro stochastic approximation algorithm (Robbins and Munro 1951). This algorithm is based on the popular Gauss-Newton method where the derivatives of the stochastic objective function are either known or estimated. Unless the true models are known it is not possible to obtain the exact derivatives. Therefore, IFT uses an unbiased estimate of the derivative. The unbiasedness of the estimate ensures guaranteed convergence of the Gauss-Newton iterations. The crucial part of the IFT algorithm is the derivative estimate.

A number of other methods are available for solving unconstrained stochastic optimization problems. Kiefer-Wolfowitz stochastic approximation algorithm (Kiefer and Wolfowitz 1952) uses a finite difference approximation of the derivative. For large dimension problems all finite difference methods involve numerous function evaluations, *i.e.*, unacceptably large number of experiments. Finite difference approximation for each variable requires at least two experiments. Consequently, the number of experiments required increases with the dimensionality of the problem. Recently, a more efficient algorithm for large dimension problems was developed by Spall (1992). They perform each experiment by simultaneously varying all the variables. This allows them to approximate derivatives with all the variables by conducting only two experiments. Although the approximate derivatives obtained using this method are not as good as those obtained through finite difference methods, a number of successful applications have been reported (Kleinman 1996). This method is extremely useful when conducting experiments is expensive.

In this chapter, a stochastic optimization method based on Nelder-Meade direct search algorithm is presented. The main advantage of using this method lies in not estimating the derivatives. It has been generally observed that Nelder-Meade method performs as well as any other derivative based method in the presence of noise. This method can be applied to any closed loop performance objective function. On the other hand, the IFT works only for a particular objective function defined in the following section.

8.3 Iterative Feedback Tuning

The most important feature of IFT is its estimate of the derivative of closed loop performance objective function from the closed loop experiments. Once the derivative is obtained for a given control structure, the stochastic optimization is performed using Gauss-Newton method. For a one-degree of freedom controller two experiments are conducted at each iteration. In the following paragraphs, the IFT algorithm is briefly

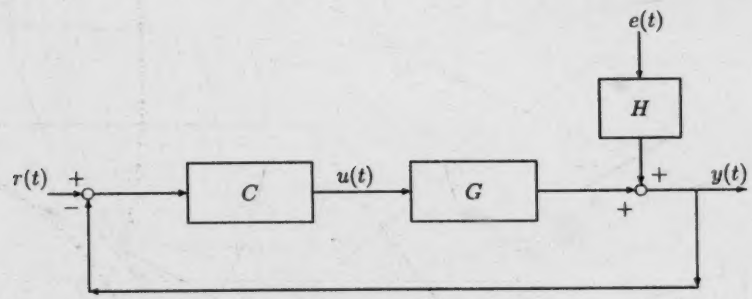


Figure 8.1: Block diagram of a Closed loop system

described.

Consider a true linear time-invariant system described by

$$\mathcal{S} \; : \; y(t) = G(q)u(t) + H(q)e(t) \tag{8.2}$$

where $G(q)$ is the true linear process model and $H(q)$ is the true linear noise model. $e(t)$ is white noise with variance, σ_e^2 and bounded fourth moments. q denotes the forward shift operator. Assume that a specific controller $C(\theta, q)$ needs to be tuned for optimal control parameters, θ. In the IFT method the following objective function is chosen for tuning the control parameters.

$$J(\theta) \;=\; \frac{1}{2N} E\left[\sum_{t=1}^{N}(\tilde{y}(t,\theta))^2 + \lambda \sum_{t=1}^{N}(u(t,\theta))^2\right] \tag{8.3}$$

where the data length is denoted by N and $\tilde{y}(t,\theta) = y(t,\theta) - r(t)$. E represents the expectation operator and λ denotes the input weighting. It is important to notice the explicit dependence of the performance objective function and the input-output data on the tunable control parameters, θ. It is of interest to tune the controller in such a way that it provides the optimal closed loop performance by minimizing $J(\theta)$. In practice, it is not possible to evaluate $J(\theta)$ due to the expectation operator. Hence, an estimate of the performance objective function

$$\hat{J}(\theta) \;=\; \frac{1}{2N}\left[\sum_{t=1}^{N}(\tilde{y}(t,\theta))^2 + \lambda \sum_{t=1}^{N}(u(t,\theta))^2\right] \tag{8.4}$$

is minimized. The justification for using this estimate arises from the fact that $\hat{J}(\theta)$ converges almost surely to $J(\theta)$ as data length tends to infinity. It is now possible to

find a minimizing θ for the above objective function using Gauss-Newton method. The controller parameter updates at iteration k are given by

$$\theta_{k+1} = \theta_k - \gamma_k R_k^{-1} \frac{\partial J}{\partial \theta}(\theta_k) \tag{8.5}$$

where R_k is some positive definite matrix, typically a Gauss-Newton approximation of the Hessian of J and the sequence γ_k must satisfy certain constraints for the algorithm to converge to a local minimum of the cost function, J (Hjalmarsson *et al.* 1998). If the derivative of J with respect to the controller parameters is available, then the above stochastic optimization problem is straightforward to solve. Since it is not easy (and in many cases not possible) to obtain an explicit expression for the derivative or to measure it directly, Hjalmarsson *et al.* (1998) proposed using an estimate that is obtained by conducting two special experiments on the closed loop for one-degree freedom controllers. An estimate of the derivative can be expressed as

$$\frac{\partial \hat{J}(\theta)}{\partial \theta} = \frac{1}{N}\left[\sum_{t=1}^{N} \tilde{y}(t,\theta)\frac{\partial \tilde{y}(t,\theta)}{\partial \theta} + \lambda \sum_{t=1}^{N} u(t,\theta)\frac{\partial u(t,\theta)}{\partial \theta}\right] \tag{8.6}$$

Since $\tilde{y}(t,\theta)$ and $u(t,\theta)$ can be measured, the IFT algorithm is implementable, if the derivatives, $\frac{\partial u(t,\theta)}{\partial \theta}$ and $\frac{\partial \tilde{y}(t,\theta)}{\partial \theta}$ can be estimated with no bias. It is possible to obtain these estimates by performing two closed loop experiments. Let $r_1(t) = r(t)$ be the reference signal during the first experiment. Then

$$y_1(t,\theta) = \frac{C(q,\theta)G(q)}{1+C(q,\theta)G(q)} r(t) + \frac{1}{1+C(q,\theta)G(q)} H(q)e_1(t) \tag{8.7}$$

where $y_1(t,\theta)$ was the output obtained from the first experiment. In the second experiment a new reference signal defined by $r_2(t) = r(t) - y_1(t,\theta)$ is used.

$$y_2(t,\theta) = \frac{C(q,\theta)G(q)}{1+C(q,\theta)G(q)} r_2(t) + \frac{1}{1+C(q,\theta)G(q)} H(q)e_2(t) \tag{8.8}$$

$e_1(t)$ and $e_2(t)$ are two different white noise sequences. In Hjalmarsson *et al.* (1998) the following unbiased estimates of the above derivatives were obtained

$$\widehat{\frac{\partial y(t,\theta)}{\partial \theta}} = \frac{1}{C(q,\theta)} \frac{\partial C(q,\theta)}{\partial \theta} y_2(t,\theta) \tag{8.9}$$

$$\widehat{\frac{\partial u(t,\theta)}{\partial \theta}} = \frac{1}{C(q,\theta)} \frac{\partial C(q,\theta)}{\partial \theta} u_2(t,\theta) \tag{8.10}$$

where $u_2(t,\theta)$ are inputs from the second input. Substituting (8.9) and(8.10) into (8.6) the performance derivative can be estimated. It has been proved that the estimate is

unbiased, *i.e.*,

$$E\left[\frac{\partial \hat{J}(\theta)}{\partial \theta}\right] = \frac{\partial J}{\partial \theta} \tag{8.11}$$

The Iterative Feedback Tuning algorithm can be viewed as a nonlinear feedback controller with the controller parameters as the input and the achieved performance as the output. A block diagram of the IFT algorithm is shown in fig.8.2.

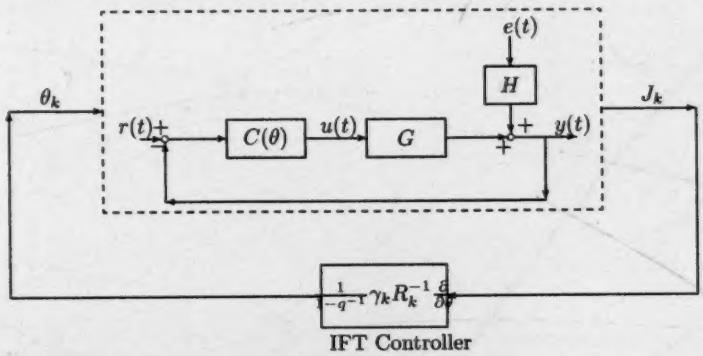


Figure 8.2: Block diagram of a Closed loop system

8.4 Model Free Controller Tuning

A modified Nelder-Meade method that is suitable for solving the stochastic optimization problem described in the previous sections is presented here. The IFT method presented in the previous section has a number of disadvantages :

- Its convergence can be slow for small signal to noise ratios.
- The derivative estimate is based on the assumption that the true process model is linear.
- The derivative estimate is valid only for the objective function defined in the previous section
- Each iteration involves at least two experiments.

Since the convergence is going to be slow for small signal to noise ratios, it is possible to use direct search methods like Nelder-Meade to obtain the same or better percentage of improvement in the closed loop performance for equal number of experiments. Nelder-Meade method involves no derivatives and as a result it can handle minor nonlinearities

in the process. Moreover, this method can be applied to any objective function. For instance, the proposed method is applicable to design model reference controllers.

These advantages of the proposed method come at a price. Even though, Nelder-Meade has been popular among process engineers there are no concrete theoretical results to prove the convergence of this algorithm. Given all the inefficiencies and failures of the Nelder-Meade algorithm (Torczon 1989, Lagarias *et al.* 1998), one might wonder why it is used at all. In many applications, for instance in industrial process control, one simply wants to find controller tunings that can provide better performance. Nelder-Meade method, generally, provides significant improvement in the first few iterations. In problems where function evaluation is expensive and function derivatives are not available, Nelder-Meade method is very effective. In the words of Lagarias *et al.* (1998) *"When it succeeds, Nelder-Meade method tends to require substantially fewer function evaluations than its alternatives, and its relative "best efficiency" often outweighs the lack of convergence theory."*. Moreover, this method is appealing due to the simplicity of its steps.

A modified Nelder-Meade algorithm is described below to solve the problem described in the previous sections. A real valued stochastic objective function $J(\theta)$ (performance measure) needs to be minimized where $\theta \in \mathbb{R}^n$. θ is a vector of controller parameters. Original Nelder-Meade method involves four parameters that define the search search directions. These parameters are called : a) reflection (ρ) b) expansion (χ) c) contraction (γ) and d) shrinkage (ω). A new parameter, σ, that takes into account the variance of the objective function is defined in this chapter. This parameter is useful in choosing the initial guess and also in subsequent Nelder-Meade iterations. It helps in faster convergence of the iterations for large signal to noise ratios. The above parameters are expected to satisfy

$$\rho > 0, \qquad \chi > 0, \qquad \chi > \rho, \qquad 0 < \gamma < 1, \qquad \sigma > 0 \quad \text{and} \quad 0 < \omega < 1. \tag{8.12}$$

In any *standard* Nelder-Meade algorithm the following choices are used for the above parameters

$$\rho = 1, \qquad \chi = 2, \qquad \gamma = \frac{1}{2}, \qquad \omega = \frac{1}{2}. \tag{8.13}$$

The new parameter introduced in this chapter, σ, takes a value that determines the speed of convergence of the algorithm. Generally, it is suggested that this value be made as close as possible to the standard deviation of the objective function. It may not always be possible to estimate this parameter accurately. An intelligent guess from the process experience is useful in fixing this parameter.

8.4.1 Algorithm

In this algorithm the equality sign is redefined using the variance parameter, σ. The function, $J(\theta)$ evaluated at $\theta^{(i)}$ and $\theta^{(j)}$ are said to be equal if and only if

$$|J(\theta^{(i)}) - J(\theta^{(j)})| \leq \sigma. \tag{8.14}$$

Note that this clearly includes the traditional definition of equality sign. For convenience, the following new symbols are defined: The equation (8.14) is alternatively represented by

$$J(\theta^{(i)}) \stackrel{\sigma}{=} J(\theta^{(j)}) \tag{8.15}$$

Similarly, it is possible to define *less than, greater than, less than or equal to and greater than or equal to* symbols

$$J(\theta^{(i)}) < J(\theta^{(j)}) - \sigma \quad \text{represented by} \quad J(\theta^{(i)}) \stackrel{\sigma}{<} J(\theta^{(j)}) \tag{8.16}$$

$$J(\theta^{(i)}) > J(\theta^{(j)}) + \sigma \quad \text{represented by} \quad J(\theta^{(i)}) \stackrel{\sigma}{>} J(\theta^{(j)}) \tag{8.17}$$

$$J(\theta^{(i)}) \leq J(\theta^{(j)}) + \sigma \quad \text{represented by} \quad J(\theta^{(i)}) \stackrel{\sigma}{\leq} J(\theta^{(j)}) \tag{8.18}$$

$$J(\theta^{(i)}) \geq J(\theta^{(j)}) - \sigma \quad \text{represented by} \quad J(\theta^{(i)}) \stackrel{\sigma}{\geq} J(\theta^{(j)}) \tag{8.19}$$

Initialization: In the Matlab function, *fminsearch*, an initial nondegenerate simplex is obtained by perturbing each element of the guess vector by 5% for nonzero elements and by 0.025% for zero elements. If the signal to noise ratio is small, depending on the objective function, it may be necessary to use larger perturbations. Hence, the initial guess is perturbed by $0.05\frac{1}{1-\sigma}$ for nonzero elements and by $0.00025\frac{1}{1-\sigma}$ for zero elements. In this chapter a simplex is called σ-non-degenerate if it is nondegenerate with respect to the equality sign defined in this section. If the objective function is "flat" or if the signal to noise ratio is large (*i.e.*, σ is large), it may be necessary to use larger perturbations in the initial guess vector to obtain a σ-nondegenerate simplex. Observe that with the definitions in this section, any two realizations of the objective function, $J(\theta)$ satisfy reflexive and symmetric properties but not the transitivity property of real numbers. These newly defined operators also satisfy the trichotomy property of real numbers.

kth iteration: At the beginning of the kth iteration a σ-nondegenerate simplex, Δ_k is given with $n+1$ vertices. Each vertex is a point in $\mathbb{R}^n$. At each iteration, k, the vertices are labelled as $\theta^k_{(1)}, \cdots, \theta^k_{(n+1)}$ such that

$$J(\theta^k_{(1)}) \stackrel{\sigma}{\leq} J(\theta^k_{(2)}) \stackrel{\sigma}{\leq} \cdots \stackrel{\sigma}{\leq} J(\theta^k_{(n+1)}) \tag{8.20}$$

At the end of this iteration a new simplex, Δ_{k+1} is defined using the $n+1$ vertices generated by kth iteration. Various steps involved in Nelder-Meade algorithm with the equality operator defined above are presented here:

a) **Reflect**: Compute the reflection point θ_r from

$$\theta^k_{(r)} = \theta^k_{(c)} + \rho(\theta^k_{(c)} - \theta^k_{(n+1)}) \tag{8.21}$$

where $\theta^k_{(c)} = 1/n \sum_{j=1}^{n} \theta^k_{(j)}$ is the centroid of the n best points. If $J(\theta^k_{(1)}) \overset{\sigma}{\leq} J(\theta^k_{(r)}) \overset{\sigma}{\leq} J(\theta^k_{(n)})$, accept $\theta^k_{(r)}$ and reject $\theta^k_{(n+1)}$ terminating the kth iteration.

b) **Expand**: If $J(\theta^k_{(r)}) \overset{\sigma}{<} J(\theta^k_{(1)})$ then evaluate the expansion point $\theta^k_{(e)}$ through

$$\theta^k_{(e)} = \theta^k_{(c)} + \chi(\theta^k_{(r)} - \theta^k_{(c)}). \tag{8.22}$$

If $J(\theta^k_{(e)}) \overset{\sigma}{<} J(\theta^k_{(r)})$, accept $\theta^k_{(e)}$ and terminate the iteration; otherwise accept $\theta^k_{(r)}$ and terminate the iteration.

c) **Contract**: If $J(\theta^k_{(r)}) \overset{\sigma}{\geq} J(\theta^k_{(n)})$ then

(i) **Outside**: If $J(\theta^k_{(n)}) \overset{\sigma}{\leq} J(\theta^k_{(r)}) \overset{\sigma}{<} J(\theta^k_{(n+1)})$ perform an outside contraction by evaluating

$$\theta^k_{(co)} = \theta^k_{(c)} + \gamma(\theta^k_{(r)} - \theta^k_{(c)}) \tag{8.23}$$

If $J(\theta^k_{co}) \overset{\sigma}{<} J(\theta^k_{(r)})$, then accept θ^k_{co} and terminate the iteration.

(ii) **Inside**: If $J(\theta^k_r) \overset{\sigma}{\geq} J(\theta^k_{(n+1)})$ an inside contraction is performed by evaluating

$$\theta^k_{(ci)} = \theta^k_{(c)} - \gamma(\theta^k_{(c)} - \theta^k_{(n+1)}) \tag{8.24}$$

If $J(\theta^k_{(ci)}) \overset{\sigma}{<} J(\theta^k_{(n+1)})$ accept $\theta_{(ci)}$ and terminate the iteration.

d) **Shrink**: Evaluate $J(\theta)$ at the n points $\beta^k_{(i)} = \theta^k_{(1)} + \omega(\theta^k_{(i)} - \theta^k_{(1)})$ for $i = 2, \cdots, n+1$. Then the vertices of the simplex, Δ_{k+1}, are given by $\theta^k_{(1)}, \beta^k_{(2)}, \cdots, \beta^k_{(n+1)}$.

Termination: Standard termination criteria like function tolerance, number of function evaluations can be used along with σ-degeneracy of the the simplex at each iteration.

The introduction of the parameter, σ, is useful in faster convergence of the algorithm. The idea behind using this parameter is to treat two controllers (or control parameters) whose performance differs by less than σ equally. Since the aim of any iterative controller tuning method is to improve the current closed loop performance, the true optimal controller is not of primary interest. As long as the proposed method provides better performance in a limited number of iterations it is useful. The variable σ, ensures that the iterations quickly move to a controller whose performance is at least σ units better.

8.5 Simulation Examples

Example 8.5.1 In this example a flexible transmission system proposed in Hjalmarsson *et al.* (1995) as a benchmark problem is used for controller design. This flexible transmission consists of three horizontal pulleys connected by two elastic belts (see fig.8.3). The input to the system is the angular position of the first pulley and the system output is the angular position of the third pulley. The control objective is to make the angular position of the third pulley as close as possible to the angular position of the first one. The dynamic behavior of the system with the above input and output can be described by the following transfer function,

$$G(q) = \frac{0.28261q + 0.50666}{q^4 - 1.41833q^3 + 1.58939q^2 - 1.31608q + 0.88642} \tag{8.25}$$

The control objective is to achieve a closed loop step response that is as close as possible to that of

$$M(q) = \frac{(1-\alpha)^2}{q^3 - 2\alpha q^2 + \alpha^2 q}, \qquad \alpha = \exp^{-0.05\bar{\omega}}, \quad \bar{\omega} = 10 \tag{8.26}$$

where $\bar{\omega}$ is the desired bandwidth. The following class of controllers are considered for optimization

$$C(q,\theta) = \frac{\theta_0 + \theta_1 q^{-1} + \theta_2 q^{-2} + \theta_3 q^{-3} + \theta_4 q^{-4} + \theta_5 q^{-5}}{1 - q^1} \tag{8.27}$$

where $\theta = [\theta_1 \;\; \theta_2 \;\; \cdots \;\; \theta_5]$. The performance of the closed loop system is measured using the following objective function with $N = 500$

$$\hat{J} = \frac{1}{N}\left[\sum_{t=1}^{N}(y(t,\theta) - M(q)r(t))^2 + \lambda\sum_{t=1}^{N}u(t,\theta)^2\right] \tag{8.28}$$

A plot showing the ratio of the magnitudes of $\hat{J}$ with the proposed method and the IFT tuning algorithm for various signal to noise ratios is shown in fig.8.4. The number of experiments in both the cases is restricted to ten. It is easy to notice that for the same number of experiments, the decrease in the objective function using the proposed method is much larger than the decrease using IFT for small signal to noise ratios.

Example 8.5.2 The simulation example given in Forssell and Ljung (2000*a*) is used here with a valve that has backlash. The process model is described by

$$G(q) = \frac{q^{-1}}{1 - 1.2q^{-1} + 0.61q^{-2}} \tag{8.29}$$

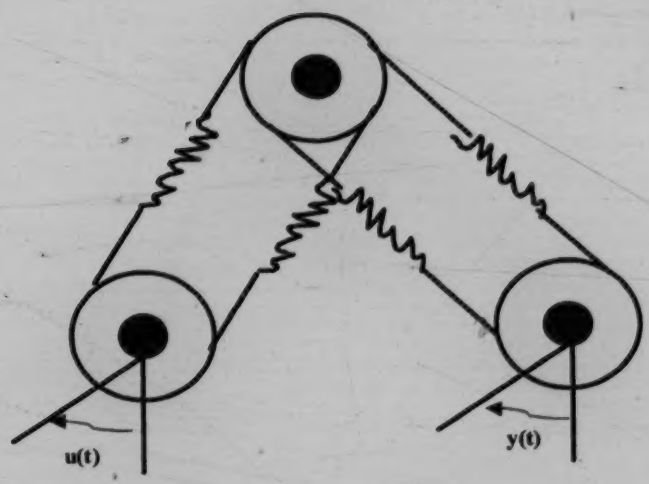


Figure 8.3: Flexible Transmission System

and the bandwidth of the backlash element is 0.1. The following PI controller is tuned using the proposed method

$$C(q) = \frac{\theta_0 + \theta_1 q^{-1}}{1 - q^{-1}} \tag{8.30}$$

An unstable initial guess, $\theta^0 = [0.3 \quad 0.1]$ lead to a stable final optimal controller tunings $\theta^f = [-0.0075 \quad 0.1119]$ for the objective function in (8.28) with $\lambda = 1$ and the data length, $N = 500$. A plot of the closed loop step response is shown in fig.8.5. The original IFT method did not work on this example with the initial guess used here. There are two possible reasons - it is either due to the initial unstable guess or the algorithm itself diverges.

8.6 Conclusions

A new method for direct controller tuning based on the closed loop performance criterion is proposed. Unlike Iterative Feedback Tuning, this method does not depend on derivative estimates and as a result can be applied to any closed loop performance criteria. It is easy to implement. On the downside, there are no convergence proofs for this method.

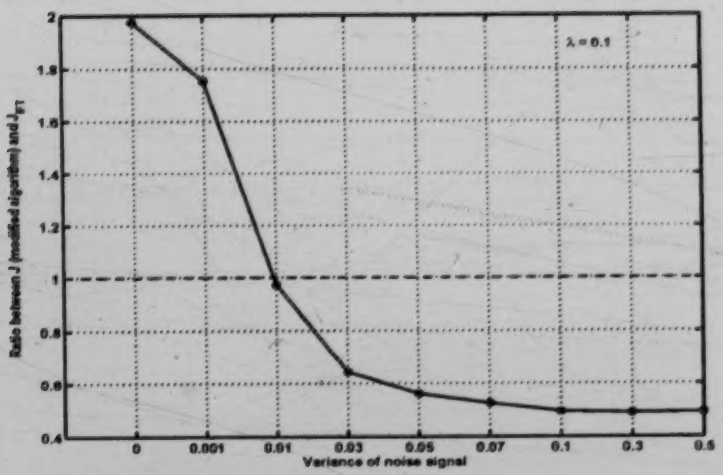


Figure 8.4: Ratio between J (modified algorithm) and J_{IFT} for noise variances $\{0;0.001;0.01;0.03;0.05;0.07;0.1;0.3;0.5\}$

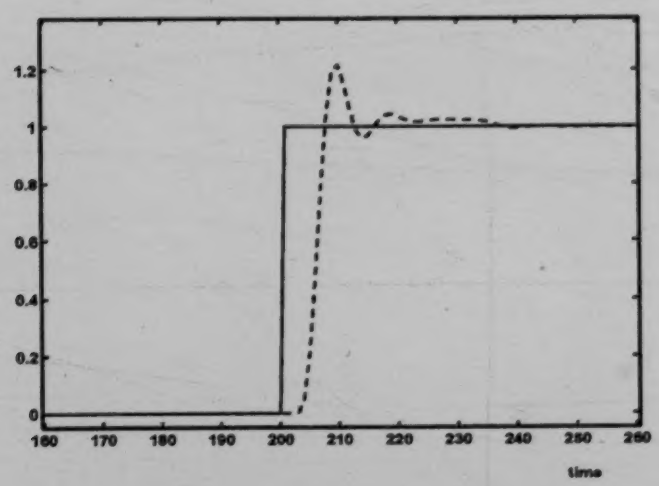


Figure 8.5: Closed loop step response of the second example

9

Nonlinear Iterative Control

Nonlinear Backstepping is a recursive design methodology which makes use of the Lyapunov stability theory. Though backstepping can be applied to a larger class of systems than other differential-geometric methods like feedback linearization, its applicability is limited to "parametric pure-feedback systems". In this work we apply the idea of backstepping to a benchmark chemical reactor by using a simple transformation of the original nonlinear model of the chemical reactor. This chemical reactor does not fall under the category of systems for which backstepping can be applied (Krstic *et al.* 1995). However, the fundamental idea involved in backstepping can still be applied to this process after a certain transformation of the original variables.

A robust adaptive nonlinear controller is also designed by introducing uncertainty into all the estimated parameters. This type of uncertainty leads to non-affine uncertain parameters which are hard to handle with the traditional backstepping algorithm. Using Lyapunov theory we derive a controller which can ensure robust stability[1].

[1] This part of research was done in collaboration with Prof. Mizumoto from Kumamoto University, Japan. Several portions of this chapter have appeared or are going to appear in the form of publications (Gopaluni *et al.* 2000, Gopaluni *et al.* 2003*f*, Mizumoto *et al.* 2002*a*, Mizumoto *et al.* 2002*b*, Mizumoto *et al.* 2002*c*)

9.1 Introduction

Nonlinearities are the rule rather than exception in chemical processes was the opening statement of the review article (Kravaris and Kantor 1990a). The fact is that virtually all processes are non-linear. Many common process control problems exhibit nonlinear behaviour, in that the relationship between the controlled and manipulated variables depends on the operating conditions. In spite of this, chemical processes are traditionally controlled by using linear system design techniques, see e.g. (Kravaris and Kantor 1990a, Henson and Seborg 1991, Kravaris 1990b, Nijmeijer 1989). The dynamic behaviour of a process can be approximated by a linear model around a certain operating point. If the process is only mildly nonlinear or remains in the vicinity of a nominal steady state, then the effects of the nonlinearities may not be severe. In that case conventional linear control design techniques are generally deemed to be adequate. A major reason to rely on linear control is the relative simplicity and computational feasibility of linear control schemes. In the recent times there has been a growing interest in nonlinear process control especially with the advent of new techniques to attack nonlinear control design problem. The most commonly used results are feedback linearization, input-output decoupling and disturbance decoupling. These topics obviously form a small portion of what can be done in the area of non-linear control, see for instance (Isidori 1995, Nijmeijer 1989).

During the last five decades there have been a number of attempts to build controllers that can stabilize and track setpoints in the presence of uncertainties in the process parameters (not necessarily time varying). The most elementary feedback loops that were designed in early 50's can tolerate significant uncertainties. With the advances in Modern Control theory (Chen 1984), it was realized that there is a need for more advanced controllers for uncertain plants to achieve optimal performance. An offshoot of this realization is the development of Adaptive Linear Control theory and the Robust Control theory. Adaptive linear control is an area where the plant model and as a result the optimal linear controller are updated continuously using online identification and control algorithms. Many results on stability and asymptotic convergence of adaptive linear controllers have been proved and a number of books have been written on adaptive linear control (Astrom and Wittenmark 1995, Goodwin and Sin 1984, Sastry and Bodson 1989, Narendra and Annaswamy 1989). Robust Control theory for linear systems was developed assuming that a frequency domain description of the uncertain bounds is known (Doyle *et al.* 1992, Zhou and Doyle 1998), but the identification of systems with frequency domain uncertainty descriptions has proved to be difficult. On

top of this, most of these techniques can handle only linear systems.

Recent advances in the use of differential-geometric tools for nonlinear feedback control (Isidori 1995, Nijmeijer 1989) has spurred a renewed interest in the adaptive control of nonlinear systems with uncertain parameters. Differential geometric methods provided us with a number of aesthetically pleasing results like feedback linearization. But, it soon became clear that these methods can not handle uncertain/unknown parameters. This lead the researchers to take recourse to some of the old techniques of proving stability using Lyapunov functions. Lyapunov function based controller design has, historically, played a very important role in nonlinear control design. Until recently its applicability was restricted to linear plants with relative degree one or two. A new recursive design procedure called *backstepping* has removed this limitation and paved the way for designing controllers for higher order nonlinear systems (Khalil 1992, Krstic *et al.* 1995).

Backstepping uses a systematic approach for the construction of feedback control laws as well as Lyapunov functions. This removes the difficulties encountered in obtaining a control Lyapunov function for higher order systems. Backstepping builds into the system a number of strong properties of global or local stability and tracking in a number of recursive steps, which are never greater than the system order. While feedback linearization methods require precise models and often cancel useful nonlinearities, backstepping designs offer a choice of design tools for handling uncertain nonlinearities and can avoid wasteful cancellation of nonlinearities (Krstic *et al.* 1995).

Inspite of its many advantages, backstepping has some drawbacks in that this method can not be applied to any general nonlinear system (the largest class of systems for which this method can be applied is not yet known). It is well known that backstepping can be applied to "parametric pure feedback systems" and "parametric strict feedback systems" with either the *matching condition* or the *extended matching condition*. Adaptive backstepping can result in over parametrization but in some cases this problem can be avoided by using *tuning functions*.

In this study the backstepping method is applied to a model of a continuous stirred tank reactor (CSTR) described in (Engell and Klatt 1993) and in (Chen *et al.* 1995). This reactor has been designated as a benchmark problem for nonlinear control system design in (Chen *et al.* 1995). This CSTR problem exhibits a number of interesting features like poor performance of the linearized model, unstable zero dynamics. It is also a process with great practical importance. A dynamic state feedback controller that achieves input-output feedback linearization was designed for the same CSTR in (Sanchez and H.Nijmeijer 1995).

This chapter is organized as follows: section 9.2 gives a detailed description of the process. In section 9.3 the control problem is posed and the non-applicability of static feedback linearization to this particular system is shown. Backstepping controller design methodology is then discussed in section 9.4. A modified adaptive backstepping algorithm is presented in section 9.5. Finally, a robust adaptive controller is derived in section 9.6. Simulation results and some explanations for the results are presented in section 9.7. The last section, section 9.8, gives conclusions.

9.2 Process Model Description

In many chemical processes, the main reaction which yields the desired product is accompanied by consecutive and parallel reactions which produce undesired by-products. An example for a general reaction scheme of this type was used by Kantor (Kantor 1986) to discuss the applicability of extended linearization techniques to the control of reactions in continuous stirred tank reactors. The same reaction mechanism was used in (Engell and Klatt 1993) to design a gain scheduling controller. This reaction mechanism was attributed to van de Vusse (van de Vusse 1964) and can be written as

$$\mathrm{A} \xrightarrow{k_1} \mathrm{B} \xrightarrow{k_2} \mathrm{C}$$

$$2\mathrm{A} \xrightarrow{k_3} \mathrm{D}$$

where A is the reactant, B the desired product, C and D are the unwanted by-products. In general, we try to make the reaction rates k_2 and k_3 small in comparison to k_1 by an appropriate choice of catalyst and reaction conditions. However, if this is not possible or only possible to a certain extent, the concentration of B in the product stream which leaves the reactor may be controlled by the inflow to the reactor and/or the reaction temperature.

The process considered here involves production of cyclopentenol (B) from cyclopentadiene (A) by acid catalysed electrophilic addition of water in dilute solution. Due to the strong reactivity of A and the product, dicyclopentadiene (D) is produced by Diels-Alder reaction as a side product, and cyclopentanediol (C) as a consecutive product by addition of another water molecule (Engell and Klatt 1993). The complete reaction scheme is

$$\mathrm{C_5H_6} \xrightarrow{+H_2O/H^{\oplus}} \mathrm{C_5H_7OH} \xrightarrow{H_2O/H^{\oplus}} \mathrm{C_5H_8(OH)_2}$$

$$2\mathrm{C_5H_6} \longrightarrow \mathrm{C_{10}H_{12}}$$

The dynamics of the reactor can be described by the following set of differential equations (see (Chen *et al.* 1995) for further details about modelling and physico-chemical properties):

$$\dot{c}_A = \frac{\dot{V}}{V_R}(c_{A0} - c_A) - k_1(\nu)c_A - k_3(\nu)c_A^2 \tag{9.1a}$$

$$\dot{c}_B = -\frac{\dot{V}}{V_R}c_B + k_1(\nu)c_A - k_2(\nu)c_B \tag{9.1b}$$

$$\dot{\nu} = \frac{\dot{V}}{V_R}(\nu_0 - \nu) + C_1(\nu_K - \nu) - \frac{1}{\rho C_\rho}(k_1(\nu)c_A d_1 + k_2(\nu)c_B d_2 + k_3(\nu)c_A^2 d_3) \tag{9.1c}$$

$$\dot{\nu}_K = \frac{1}{m}\dot{Q}_K + \frac{C_2}{m}(\nu - \nu_K) \tag{9.1d}$$

The concentrations of A and B are c_A and c_B respectively ($c_A \geq 0$, $c_B \geq 0$). The temperature in the reactor is denoted by ν while the temperature in the cooling jacket is given by ν_K, both temperatures are expressed in absolute degrees (deg K). The reaction velocities are assumed to depend on the temperature via the Arrhenius law:

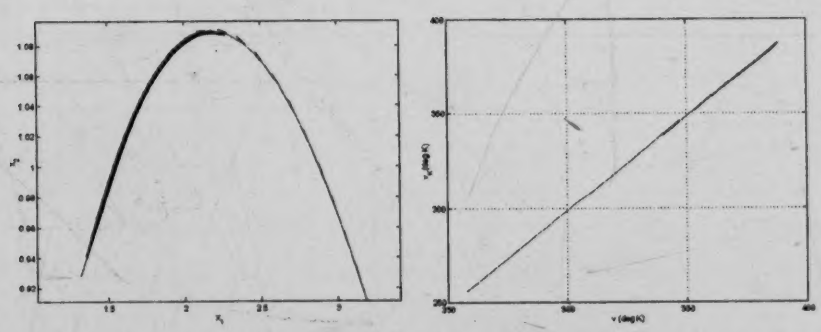
$$k_i(\nu) = k_{i0}.exp(\frac{E_i}{\nu}), \quad i = 1, 2, 3 \tag{9.2}$$

The inflow of the reactor is composed only of substance A with inflow concentration and temperature given by c_{A0} and ν_0, respectively. Values for the physical and chemical parameters are given in Table 9.1. Before we proceed with the formulation of our control problem, we will take a look at the local or approximated steady state map of this system. Steady state map of this system is shown in fig.9.1(a) & 9.1(b). The steady state map between the concentrations of c_A and c_B is an inverted parabola and that between the temperature in the jacket and the temperature in the reactor is a straight line.

9.3 Feedback Linearization of CSTR

We proceed with the formulation of the control problem. We want to regulate the concentration of product B and simultaneously regulate the ratio c_B/c_A. We can interpret this ratio as a measure of the effectiveness of the reaction and certainly this ratio is useful for regulating the performance of the reactor. We make the following choice of state variables :

$$x = [c_A \quad c_B \quad \nu \quad \nu_K]^T \tag{9.3}$$



(a) Approximate steady state map of the system - c_A vs c_B

(b) Approximate steady state map of the system - ν vs ν_k

Figure 9.1: Steady state map of the CSTR

Table 9.1: Process Parameters

Symbol	*Values of parameters*		*Symbol*	*Values of parameters*	
k_{10}	$3.575.10^8$	sec^{-1}	d_3	-41.85	$kJ/mol\ A$
k_{20}	$3.575.10^8$	sec^{-1}	ρ	0.9342	kg/l
k_{30}	$2.512.10^6$	$mol\ A.sec^{-1}$			
E_1	-9758.3	$\deg K$	C_p	3.01	$kJ/kg.K$
E_2	-9758.3	$\deg K$	C_1	$85.6347.10^{-4}$	sec^{-1}
E_3	-8560	$\deg K$	C_2	0.2408	$kJ/sec.K$
d_1	4.20	$kJ/mol\ A$	m	10.0	kJ/K
d_2	-11.0	$kJ/mol\ B$	V_R	0.01	m^3

Under this choice of state variables the dynamics of the system takes the form

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \tag{9.4}$$

where $u_1 = \frac{\dot{V}}{V_R}$ and $u_2 = \dot{Q}_K$ are the manipulated variables, the vector field $f(x)$ is given by

$$f(x) = \begin{bmatrix} -k_1(x_3)x_1 - k_3(x_3)x_1^2 \\ k_1(x_3)x_1 - k_2(x_3)x_2 \\ C_1(x_4 - x_3) - \frac{1}{\rho C_\rho} f_{x_3} \\ \frac{C_2}{m}(x_3 - x_4) \end{bmatrix} \tag{9.5}$$

where

$$f_{x_3} = k_1 x_1 d_1 + k_2 x_2 d_2 + k_3 x_1^2 d_3 \tag{9.6}$$

and the vector fields $g_1(x)$ and $g_2(x)$ are given by

$$g_1(x) = \begin{bmatrix} (c_{A0} - x_1) \\ -x_2 \\ (\nu_0 - x_3) \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} \tag{9.7}$$

and the output functions are

$$y_1(x) = h_1(x) = x_2\,, \quad y_2(x) = h_2(x) = \frac{x_2}{x_1} \tag{9.8}$$

We also assume that the state variables satisfy the following conditions:

$$x_1 > 0\ ,\ x_2 > 0\ ,\ x_3 > 0 \tag{9.9}$$

As a first step the relative degree of this system needs to be calculated to check the possibility of simplifying the model equations through some coordinate transformation. Since this is a MIMO system we need to define the relative degree for MIMO systems.

Definition (Isidori 1995): Consider a MIMO nonlinear system of the form

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j \tag{9.10}$$

$$y_i = h_i(x) \quad i = 1, 2, ..., p \tag{9.11}$$

The nonlinear system is said to have *vector relative degree* $r_1, r_2, ..., r_m$ at the point x_0 if:

1. $L_{g_j} L_f^k h_i(x) = 0$ for all $1 \leq i\ ,\ j \leq m$, for all $k < r_i - 1$, and for all x in a neighborhood of x_0.
2. The $m \times m$ *decoupling matrix*

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{r_m - 1} h_m(x) \end{bmatrix} \tag{9.12}$$

is nonsingular at the point x_0. The integer r_i represents the smallest relative degree of the i^{th} output with respect to any of the m inputs. Nonsingularity of the decoupling matrix $A(x)$ may be viewed as the MIMO generalization of the condition that $L_g L_f^{r-1}(x_0) \neq 0$. This is a necessary and sufficient condition for the solution of the input-output decoupling control problem with static state feedback.

The characteristic numbers or the relative degree of the system (9.4) are $r_1 = 1$ and $r_2 = 1$, since $L_{g_1} h(x) \neq 0$ and $L_{g_2} h(x) = 0$ for both the outputs. Therefore, after differentiating the outputs once, the following equation is obtained:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = F(x) + G(x)u \tag{9.13}$$

where $u = [u_1 \ \ u_2]^T$ and the matrix $G(x)$ is the decoupling matrix, which is given by

$$G(x) = \begin{bmatrix} -x_2 & 0 \\ -c_{A0} \frac{x_2}{x_1^2} & 0 \end{bmatrix} \tag{9.14}$$

Since the decoupling matrix is singular we can conclude that the original system (9.4) can not be input-output decoupled by means of static feedback (Nijmeijer 1989). In (Sanchez and H.Nijmeijer 1995) a dynamic state feedback controller was used to input-output linearize the system. It was also shown that the compensated system is equivalent to a linear system, or, in otherwords has a trivial zero dynamics (Isidori 1995, Nijmeijer 1989). Their method does not take into account the boundedness of inputs, outputs and state signals and parameter and model uncertainties. In fact, their simulation results clearly show that the inputs, inlet flow rate and the heat removal rate, are too large in the vicinity of a step change in the set points. Moreover, feedback linearization can not guarantee either asymptotic tracking or global stability in presence of uncertain parameters. As described earlier, the natural choice of controller which can address all the above issues is a controller designed using the idea of backstepping. Before we proceed with the formulation of the backstepping problem for this system, we note that this system can be input-output linearized by a different choice of outputs. For *e.g.*, if

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{9.15}$$

The decoupling matrix is given by

$$D^1(x) = \begin{bmatrix} -x_2 & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \tag{9.16}$$

which is clearly nonsingular and therefore the system can be input-output linearized by means of a static state feedback (Nijmeijer 1989).

9.4 Backstepping Controller Design

Major advances in the differential-geometric theory of nonlinear feedback control in the mid-1980's has given a fillip to the research in adaptive nonlinear control. A comprehensive treatment of this theory is given in the books by (Isidori 1995) and (Nijmeijer 1989). But it soon became clear that along with its many advantages, differential-geometric methods have some shortcomings. They are found to be very sensitive to unknown/uncertain parameters. This first led to the development of adaptive nonlinear controllers for systems which satisfy *matching* condition. The first such adaptive scheme by (Craig 1988) required joint acceleration measurements, but this impractical assumption was soon removed by (Slotine and Li 1987, Slotine and Li 1988, Middleton and Goodwin 1988, Ortega and Spong 1989) and others. A more general treatment of adaptive nonlinear regulation has been given by (Taylor *et al.* 1989). The *matching* condition was later on relaxed to the *extended matching* condition (Kanellakopoulos *et al.* 1991). Adaptive backstepping was developed by Kanellakopoulos *et al* (Kanellalopoulos 1991) in collaboration with Peter Kokotovic and Steve Morse. The problem of over parametrization with the backstepping method was solved when Krstic (Krstic 1994) introduced tuning functions.

Backstepping is a popular new controller design technique for uncertain plants with a certain model structure. Though the general proofs are restricted to plants with some special structures, the same idea can be used for other structures too. The largest general structure of the system on which backstepping can be applied is not yet known. Though the idea of backstepping is new it is based on well known concepts of Lyapunov function and control Lyapunov functions. Backstepping simplifies the problem of finding a control Lyapunov function (clf) for a multi state system into that of finding the clf for a subsystem with a fewer number of states (usually one) by creating a number of virtual control signals.

It has been shown that state feedback backstepping can be applied to systems of the so-called class of "parametric pure-feedback systems" and to its subclass "parametric strict-feedback systems". The class of pure-feedback systems with unknown parameters

is well represented by the following third order system (Krstic *et al.* 1995)

$$\begin{aligned} \dot{x_1} &= x_2 + \varphi_1^T(x_1, x_2)\theta \\ \dot{x_2} &= x_3 + \varphi_2^T(x_1, x_2, x_3)\theta \\ \dot{x_3} &= u + \varphi_3^T(x_1, x_2, x_3)\theta \end{aligned} \tag{9.17}$$

where $p \times 1$ vector θ is constant and unknown. The pure feedback systems are characterized by the affine dependence of the θ and the structure of known nonlinearities φ_1, φ_2 and φ_3. The function φ_1 must not depend on x_3 and it should depend on x_2, likewise, φ_2 must depend on x_3. The particular CSTR that we are going to study in this project does not fall into this class of systems, but the same idea can be used.

9.4.1 Problem formulation

Using the same notation as in Section 9.3 for the states the nonlinear system can be represented as follows

$$\dot{x}_1 = u_1(c_{A0} - x_1) - k_1(x_3)x_1 - k_3(x_3)x_1^2 \tag{9.18a}$$

$$\dot{x}_2 = -u_1 x_2 + k_1(x_3)x_1 - k_2(x_3)x_2 \tag{9.18b}$$

$$\dot{x}_3 = u_1(\nu_0 - x_3) + C_1(x_4 - x_3) - \frac{1}{\rho C_\rho}(k_1(x_3)x_1 d_1 + k_2(x_3)x_2 d_2 + k_3(x_3)x_1^2 d_3) \tag{9.18c}$$

$$\dot{x}_4 = \frac{1}{m}u_2 + \frac{C_2}{m}(x_3 - x_4) \tag{9.18d}$$

Where $k_i(x_3) = k_{i0}e^{\frac{E_i}{x_3}}$. k_{i0} and E_i are constants. For the system (9.18a-9.18d) there is only one disturbance (ν_0) and there are two inputs. The control objective is to let the two outputs track two separate reference trajectories with bounded inputs. It is important to note that we can not take the system to any arbitrary state (x_1, x_2). This is clear from figs.9.1(a), 9.1(b), no choice of inputs can take the system to a point inside or outside the parabola.

9.4.2 Backstepping Controller

The system (9.18a-9.18d) does not have the "parametric pure feedback" structure, but the idea of backstepping can still be applied to this system by a judicious choice of virtual inputs. Eqn (9.18a) contains two states along with an actual input. Let us

assume that all the states are available from measurements (state feedback). Let the reference trajectory for the first state, x_1, be x_{1r}. The error system can be written as

$$\dot{z}_1 = \dot{x}_1 - \dot{x}_{1r} = u_1(c_{A0} - x_1) - k_1(x_3)x_1 - k_3(x_3)x_1^2 - \dot{x}_{1r} \tag{9.19}$$

where $z_1 = x_1 - x_{1r}$. It is assumed that the first derivative of the reference signal exists and is well defined. Then the following choice of input, u_1 and the control Lyapunov function V_0 would render the subsystem (9.19) globally asymptotically stable.

$$u_1 = \frac{-c_1 z_1 + k_1 x_1 + k_3 x_1^2 + \dot{x}_{1r}}{c_{A0} - x_1} \tag{9.20}$$

$$V_0 = \frac{1}{2} z_1^2 \tag{9.21}$$

where c_1 is a positive constant.

Note that the denominator of the first input, u_1, can never be zero and therefore the input remains finite though it may not be within practically feasible limits. Now substituting the input into the error equation (9.19), we get

$$\dot{z}_1 = -c_1 z_1$$

Therefore,

$$\begin{aligned} \dot{V}_0 &= z_1 \dot{z}_1 \\ &= -c_1 z_1^2 \end{aligned} \tag{9.22}$$

$\Rightarrow$

$$\dot{V}_0 < 0 \quad \forall \; c_1 > 0, \; z_1 \neq 0$$

As a result, the subsystem (9.18a) is globally asymptotically stable. Now, we take a step back and consider the error system for the second state

$$\dot{z}_2 = \dot{x}_2 - \dot{x}_{2r} = -u_1 x_2 + k_1(x_3)x_1 - k_2(x_3)x_2 - \dot{x}_{2r} \tag{9.23}$$

Since we have already chosen the first input, u_1, we don't have any leverage over that input. We can not manipulate x_1 or we can not take this state to an arbitrary value since it has to follow a trajectory. That leaves us with only one possibility, *i.e.*, take the third state, x_3, to a desired value such that x_2 asymptotically tracks the reference signal (x_{2r}). But the third state, reactor temperature, can not be manipulated directly. It can only be changed indirectly by manipulating the second input, u_2. The virtual control x_{3r} which lets the second state track its reference trajectory is called a stabilizing function.

Our aim is to augment the original Lyapunov function V_0 with the second error variable and find stabilizing trajectory for x_3. The following trajectory for x_3 makes the subsystem (9.19,9.23) globally asymptotically stable

$$x_{3r} = \frac{E_1}{ln\left[\frac{-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}}{k_{10}x_1 - k_{20}x_2}\right]} \tag{9.24}$$

where c_2 is a positive constant. This simplification is possible because $E_1 = E_2$. Now the augmented Lyapunov function is

$$V_1 = V_0 + \frac{1}{2}z_2^2 \tag{9.25}$$

$\Rightarrow$

$$\begin{aligned} \dot{V}_1 &= z_1\dot{z}_1 + z_2\dot{z}_2 \\ &= -c_1 z_1^2 - c_2 z_2^2 \end{aligned} \tag{9.26}$$

$\Rightarrow$

$$\dot{V}_1 < 0 \quad \forall\ c_1, c_2 > 0 \quad (z_1, z_2) \neq 0$$

The subsystem (9.19,9.23) is globally asymptotically stable with the virtual input (9.24) and the clf (9.25), but if $x_3 \neq x_{3r}$, then nothing can be said about the asymptotic stability of this subsystem. It will be shown in the following paragraphs that by making x_3 and x_4 track their respective trajectories arbitrarily closely, global asymptotic stability can be achieved.

The virtual input x_{3r} can be manipulated by adjusting another virtual input x_4 through the second input. The third error system is then

$$\begin{aligned} \dot{z}_3 &= \dot{x}_3 - \dot{x}_{3r} \\ &= u_1(\nu_0 - x_3) + C_1(x_4 - x_3) - \frac{1}{\rho C_\rho} f_{x_3} - \dot{x}_{3r} \end{aligned} \tag{9.27}$$

where f_{x_3} is defined by (9.6).

Now choose

$$x_{4r} = \frac{1}{C_1}\left[-c_3 z_3 - u_1(\nu_0 - x_3) + C_1 x_3 + \frac{1}{\rho C_\rho} f_{x_3} + \dot{x}_{3r}\right] \tag{9.28}$$

Therefore, if x_{4r} were the true input, the derivative of the following Lyapunov function will be negative definite

$$V_3 = \frac{1}{2}z_3^2 \tag{9.29}$$

$\Rightarrow$

$$\dot{V}_3 = -c_3 z_3^2 \tag{9.30}$$

But since x_{4r} is not the true input,

$$\dot{V}_3 = -c_3 z_3^2 + C_1 z_3 z_4 \tag{9.31}$$

The fourth error system is then

$$\begin{aligned} \dot{z}_4 &= \dot{x}_4 - \dot{x}_{4r} \\ &= \frac{1}{m} u_2 + \frac{C_2}{m}(x_3 - x_4) - \dot{x}_{4r} \end{aligned} \tag{9.32}$$

Choosing the input

$$u_2 = -c_4 m z_4 - C_2(x_3 - x_4) - C_1 m z_3 + m\dot{x}_{4r} \tag{9.33}$$

We get

$$\dot{z}_4 = -c_4 z_4 - C_1 z_3 \tag{9.34}$$

The subsystem (9.27,9.32) will globally asymptotically track any two trajectories x_{3r} and x_{4r} with the lyapunov function

$$V_4 = \frac{1}{2} z_3^2 + \frac{1}{2} z_4^2 \tag{9.35}$$

and its derivative

$$\dot{V}_4 = -c_3 z_3^2 - c_4 z_4^2 < 0 \quad \forall \; c_3, c_4 > 0 \quad (z_3, z_4) \neq (0,0) \tag{9.36}$$

Therefore in due course of time, whatever is the reference trajectory, x_{3r}, the errors z_3 and z_4 can be made arbitrarily small by manipulating the second input, u_2. As mentioned earlier, since x_{3r} is only a virtual input, x_3 will never be exactly x_{3r} and therefore (9.26) has to be modified. Substituting $z_3 + x_{3r}$ for x_3 in (9.23), we get

$$\dot{z}_2 = -u_1 x_2 + k_{10} e^{\frac{E_1}{z_3 + z_{3r}}} - k_{20} e^{\frac{E_2}{z_3 + z_{3r}}} - \dot{x}_{2r} \tag{9.37}$$

Note that $\dot{z}_2 \neq -c_2 z_2$ as long as $z_3 \neq 0$. But from (9.36), z_3 can be made arbitrarily small. Also, note that $\dot{z}_2$ and z_2 are bounded for any arbitrary values of z_3 and x_{3r}. It follows from the fact that u_1 is positive and $k_1(x_3)$ and $k_2(x_3)$ are bounded. Consequently, $\dot{z}_2$ can be made arbitrarily close to $-c_2 z_2$. As a result, (9.26) will be negative definite after

a finite interval of time has elapsed and it will remain negative definite thereafter. Thus, we are able to prove that with the virtual inputs (stabilizing functions) as defined above, we can achieve a globally asymptotically tracking controller. In summary, the controller is

$$\begin{aligned} u_1 &= \frac{-c_1 z_1 + k_1 x_1 + k_3 x_1^2 + \dot{x_{1r}}}{c_{A0} - x_1} \\ x_{3r} &= \frac{E_1}{ln\left[\frac{-c_2 z_2 + u_1 x_2 + \dot{x_{2r}}}{k_{10} x_1 - k_{20} x_2}\right]} \\ x_{4r} &= \frac{1}{C_1}\left[-c_3 z_3 - u_1(\nu_0 - x_3) + C_1 x_3 + \frac{1}{\rho C_p} f_{x_3} + \dot{x}_{3r}\right] \\ u_2 &= -c_4 m z_4 - C_2(x_4 - x_3) - C_1 m z_3 - m\dot{x}_{4r} \end{aligned} \tag{9.38}$$

9.4.3 Backstepping without cancelling useful nonlinearities

One major advantage of backstepping is that by a careful choice of stabilizing functions unnecessary cancellation of useful nonlinearities can be avoided (unlike feedback linearization). Replacing x_1 with $z_1 + x_{1r}$ in (9.19), we get

$$\begin{aligned} \dot{z}_1 &= -u_1 z_1 - k_1(x_3) z_1 + u_1(c_{A0} - x_{1r}) - k_1(x_3) x_{1r} \\ &\quad -k_3(x_3) x_1^2 - \dot{x}_{1r} \end{aligned} \tag{9.39}$$

Now we can choose

$$u_1 = \frac{-c_1 z_1 + k_1 x_{1r} + k_3 x_1^2 + \dot{x}_{1r}}{c_{A0} - x_{1r}} \tag{9.40}$$

which results in

$$\dot{z}_1 = -c_1 z_1 - u_1 z_1 - k_1(x_3) z_1 \tag{9.41}$$

Note that u_1, $k_1(x_3)$ are always positive. Similarly x_{3r}, x_{4r}, u_2 can be modified to

$$x_{3r} = \frac{E_1}{ln\left[\frac{-c_2 z_2 + u_1 x_{2r} + \dot{x_{2r}}}{k_{10} x_1 - k_{20} x_{2r}}\right]} \tag{9.42}$$

$$x_{4r} = \frac{1}{C_1}\left[-c_3 z_3 - u_1(\nu_0 - x_{3r}) + C_1 x_{3r} + \frac{1}{\rho C_p} f_{x_3} + \dot{x}_{3r}\right] \tag{9.43}$$

$$u_2 = -c_4 m z_4 - C_2(x_3 - x_{4r}) - C_1 m z_3 - m\dot{x}_{4r} \tag{9.44}$$

The derivatives in the virtual and true inputs result in large values for inputs. This situation can be rectified to an extent by another simplification that just drops the

derivatives in the inputs. Dropping the derivatives would still ensure tracking due to the physical nature of the process. We are only interested in tracking approximate step changes in setpoints, therefore, $\dot{x}_{1r}$and $\dot{x}_{2r}$ would go to zero very fast. Ignoring these derivatives will not cause any problems as far as tracking x_1 and x_2 are concerned. If we ignore the derivatives $\dot{x}_{3r}$ and $\dot{x}_{4r}$, the Lyapunov function V_4 has to be changed to

$$\dot{\tilde{V}}_4 = -c_3 z_3^2 - c_4 z_4^2 - \dot{x}_{3r} z_3 - \dot{x}_{4r} z_4 \tag{9.45}$$

Since $\dot{x}_{1r}$ and $\dot{x}_{2r}$ go to zero, $\dot{x}_{3r}$ would also approach zero eventually and as a result $\dot{x}_{4r}$ too approaches zero. Therefore, $\dot{\tilde{V}}_4$ would still be negative definite after a certain finite time. Also, note that if $\dot{x}_{4r}$ and z_4 have the same sign they help in reducing the error z_4. It is also an invariant set, once you are in the set where $\dot{x}_{4r}$ and z_4 have the same sign you will never leave it. If they have opposite signs they can increase the Lyapunov function and thus the errors, but the nature of the process and the output tracking requirements do not allow this and therefore $\dot{x}_{4r}$ has to change its sign after some finite time and enter the above invariant set. The arguments apply to the other term, $\dot{x}_{3r} z_3$ as well. In the rest of the analysis these derivatives are ignored since they are found to be harmless. Of course this can have a detrimental effect on the transient performance but that is the price of trying to get bounded inputs

9.5 Modified algorithm

The elegance of the backstepping approach was destroyed by the presence of the term e^{E_i/x_3} in the second ODE. Since, x_3, which is our first virtual input does not appear affine linearly in the second ODE, we had to resort to some *ad hoc* explanations for asymptotic tracking. An interesting transformation can be used to avoid this problem. Let us define a new state

$$\tilde{x}_3 = e^{\frac{E_1}{x_3}} \tag{9.46}$$

then

$$\dot{\tilde{x}}_3 = -\tilde{x}_3 \frac{[ln(\tilde{x}_3)]^2}{E_1} \dot{x}_3 \tag{9.47}$$

With this transformation the original system can be rewritten as

$$\dot{x}_1 = u_1(c_{A0} - x_1) - k_{10}\tilde{x}_3 x_1 - k_{30}\tilde{x}_3^{E_3/E_1} x_1^2 \tag{9.48}$$

$$\dot{x}_2 = -u_1 x_2 + k_{10}\tilde{x}_3 x_1 - k_{20}\tilde{x}_3 x_2 \tag{9.49}$$

$$\dot{\tilde{x}}_3 = -\tilde{x}_3 \frac{[ln(\tilde{x}_3)]^2}{E_1}\left[u_1(\nu_0 - \frac{E_1}{ln\tilde{x}_3}) + C_1(x_4 - \frac{E_1}{ln\tilde{x}_3}) - \frac{1}{\rho C_\rho} f_{x_3}\right] \tag{9.50}$$

$$\dot{x}_4 = \frac{1}{m}u_2 + \frac{C_2}{m}(\frac{E_1}{ln\tilde{x}_3} - x_4) \tag{9.51}$$

Step 1:
Choose

$$u_1 = \frac{-c_1 z_1 + k_{10}\tilde{x}_3 x_1 + k_{30}\tilde{x}_3^{E_3/E_1} x_1^2 + \dot{x}_{1r}}{c_{A0} - x_1} \tag{9.52}$$

and the Lyapunov function

$$V_0 = \frac{1}{2}z_1^2 \tag{9.53}$$

$\Rightarrow$

$$\begin{aligned} \dot{V}_0 &= z_1 \dot{z}_1 \\ &= -c_1 z_1^2 < 0 \quad \forall c_1 > 0 \quad z_1 \neq 0 \end{aligned} \tag{9.54}$$

Step 2: The second error variable is

$$\dot{z}_2 = -u_1 x_2 + k_{10}\tilde{x}_3 x_1 - k_{20}\tilde{x}_3 x_2 - \dot{x}_{2r} \tag{9.55}$$

Augment the Lyapunov function V_0

$$V_1 = V_0 + \frac{1}{2}z_2^2 \tag{9.56}$$

$\Rightarrow$

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 + z_2 \dot{z}_2 \\ &= -c_1 z_1^2 + z_2[-u_1 x_2 + k_{10}\tilde{x}_3 x_1 - k_{20}\tilde{x}_3 x_2 - \dot{x}_{2r}] \end{aligned} \tag{9.57}$$

Now choosing

$$\tilde{x}_{3r} = \frac{-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}}{(k_{10}x_1 - k_{20}x_2)} \tag{9.58}$$

and replacing $\tilde{x}_3$ by $z_3 + \tilde{x}_{3r}$ in (9.57), we get

$$\begin{aligned}\dot{V}_1 &= -c_1 z_1^2 + z_2[-u_1 x_2 + k_{10}\tilde{x}_{3r}x_1 - k_{20}\tilde{x}_{3r}x_2 - \dot{x}_{2r}] \\ &\quad + z_2 z_3(k_{10}x_1 - k_{20}x_2) \\ &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3(k_{10}x_1 - k_{20}x_2) \end{aligned} \tag{9.59}$$

where c_2 is a positive constant. $\dot{V}_1$ is clearly not negative definite. We try to cancel the indefinite term $z_2 z_3(k_{10}x_1 - k_{20}x_2)$ in the next augmentation step. Observe that the denominator of $\tilde{x}_{3r}$ is never zero (see fig.9.1(a))

Step 3: The third error variable is

$$\begin{aligned}\dot{z}_3 &= \dot{\tilde{x}}_3 - \dot{\tilde{x}}_{3r} \\ &= -\tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}\left[u_1(\nu_0 - \frac{E_1}{ln\tilde{x}_3}) + C_1(x_4 - \frac{E_1}{ln\tilde{x}_3}) - \frac{1}{\rho C_\rho}f_{\tilde{x}_3}\right] - \dot{\tilde{x}}_{3r} \end{aligned} \tag{9.60}$$

(9.61)

Augment V_1 to

$$\dot{V}_2 = V_1 + \frac{1}{2}z_3^2 \tag{9.62}$$

$\Rightarrow$

$$\begin{aligned}\dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3(k_{10}x_1 - k_{20}x_2) + z_3\dot{z}_3 \\ &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3(k_{10}x_1 - k_{20}x_2) \\ &\quad + z_3\left[-\tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}\left(u_1(\nu_0 - \frac{E_1}{ln\tilde{x}_3}) + C_1(x_4 - \frac{E_1}{ln\tilde{x}_3}) - \frac{1}{\rho C_\rho}f_{\tilde{x}_3}\right) - \dot{\tilde{x}}_{3r}\right] \end{aligned}$$

Choosing

$$\begin{aligned}x_{4r} &= \frac{1}{C_1}\left[-u_1(\nu_0 - \frac{E_1}{ln\tilde{x}_3}) + C_1\frac{E_1}{ln\tilde{x}_3} + \frac{1}{\rho C_\rho}f_{\tilde{x}_3}\right] \\ &\quad + \frac{1}{-\tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1}\left[-c_3 z_3 - z_2(k_{10}x_1 - k_{20}x_2) + \dot{\tilde{x}}_{3r}\right] \end{aligned} \tag{9.63}$$

Substituting $z_4 + x_{4r}$ for x_4, we get

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 - \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1 z_3 z_4 \tag{9.64}$$

Step 4: The fourth error variable is

$$\begin{aligned}\dot{z}_4 &= \dot{x}_4 - \dot{x}_{4r} \\ &= \frac{1}{m}u_2 + \frac{C_2}{m}(\frac{E_1}{ln\tilde{x}_3} - x_4) - \dot{x}_{4r} \end{aligned} \tag{9.65}$$

Augmenting the Lyapunov function V_2, we get

$$V_3 = V_2 + \frac{1}{2}z_4^2 \tag{9.66}$$

$\Rightarrow$

$$\begin{aligned}\dot{V}_3 &= -c_1z_1^2 - c_2z_2^2 - c_3z_3^2 - \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1z_3z_4 + z_4\dot{z}_4 \\ &= -c_1z_1^2 - c_2z_2^2 - c_3z_3^2 - \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1z_3z_4 \\ &\quad + z_4\left[\frac{1}{m}u_2 + \frac{C_2}{m}(\frac{E_1}{ln\tilde{x}_3} - x_4) - \dot{x}_{4r}\right]\end{aligned} \tag{9.67}$$

Choosing

$$u_2 = m\left[-c_4z_4 + \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1z_3 - \frac{C_2}{m}(\frac{E_1}{ln\tilde{x}_3} - x_4) + \dot{x}_{4r}\right] \tag{9.68}$$

we get

$$\dot{V}_3 = -c_1z_1^2 - c_2z_2^2 - c_3z_3^2 - c_4z_4^2 < 0 \tag{9.69}$$

$$\forall c_i > 0 \quad (z_1, z_2, z_3, z_4) \neq (0,0,0,0) \tag{9.70}$$

In summary the controller is

$$\begin{aligned}u_1 &= \frac{-c_1z_1 + k_{10}\tilde{x}_3x_1 + k_{30}\tilde{x}_3^{E_3/E_1}x_1^2 + \dot{x}_{1r}}{c_{A0} - x_1} \\ \tilde{x}_{3r} &= \frac{-c_2z_2 + u_1x_1 + \dot{x}_{2r}}{(k_{10}x_1 - k_{20}x_2)} \\ x_{4r} &= \frac{1}{C_1}\left[-u_1(\nu_0 - \frac{E_1}{ln\tilde{x}_3}) + C_1\frac{E_1}{ln\tilde{x}_3} + \frac{1}{\rho C_\rho}f_{\tilde{z}_3}\right] \\ &\quad + \frac{1}{-\tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1}\left[-c_3z_3 - z_2(k_{10}x_1 - k_{20}x_2) + \dot{\tilde{x}}_{3r}\right] \\ u_2 &= m\left[-c_4z_4 - C_1z_3 - \frac{C_2}{m}(\frac{E_1}{ln\tilde{x}_3} - x_4) + \dot{x}_{4r}\right]\end{aligned} \tag{9.71}$$

and the closed loop system is

$$\begin{aligned}\dot{z}_1 &= -c_1z_1 \\ \dot{z}_2 &= -c_2z_2 + z_3(k_{10}x_1 - k_{20}x_2) \\ \dot{z}_3 &= -c_3z_3 - z_2(k_{10}x_1 - k_{20}x_2) - \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1z_4 \\ \dot{z}_4 &= -c_4z_4 + \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1z_3\end{aligned} \tag{9.72}$$

or in matrix form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -c_1 & 0 & 0 & 0 \\ 0 & -c_2 & (k_{10}x_1 - k_{20}x_2) & 0 \\ 0 & -(k_{10}x_1 - k_{20}x_2) & -c_3 & -\tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1 \\ 0 & 0 & \tilde{x}_3\frac{[ln(\tilde{x}_3)]^2}{E_1}C_1 & -c_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

Like any other closed loop system with a backstepping controller, the "system" matrix has a negative diagonal and is skew symmetric.

9.5.1 Adaptive Backstepping

The power of backstepping lies not in being able to design a controller for large dimensional problems but in controlling plants with uncertain parameters. In the model (9.1), there are a number of estimated parameters. Invariably all the estimated parameters have some uncertainty associated with them. Please note that the derivatives in the following paragraphs are numerically approximated to make the simulations possible. In the next section a robust adaptive controller which takes into account uncertainties in all the estimated parameters is derived. The following four uncertain parameters (Engell and Klatt 1993) are used to derive the adaptive backstepping controller.

(a) $k_{10} = k_{20} = (3.575 \pm 1.11)10^8$

(b) $k_{30} = (2.512 \pm 0.075)10^6$

(c) $C_p = (3.01 \pm 0.04)$

(d) $\rho = (0.9342 \pm 4.0).10^{-4}$

In this section we only consider the first two uncertain parameters for the sake of simplicity in exposition. The following development of adaptive backstepping is based on the modified algorithm.

Step 1:

Let us represent the estimated values of k_{10} and k_{30} by $\tilde{k}_{10}10^8$ and $\tilde{k}_{30}10^6$, where $\tilde{k}_{10}$ and $\tilde{k}_{30}$ are the uncertain parameters. Henceforth we assume that $\tilde{k}_{10} = \tilde{k}_{20}$. A number of cancellations that we have carried out in the previous section are not possible if the parameters are uncertain. Substituting the input (9.52) with estimated parameters into $\dot{z}_1$

$$\dot{z}_1 = u_1(c_{A0} - x_1) - k_{10}10^8\tilde{x}_3x_1 - k_{30}10^6\tilde{x}_3^{E_3/E_1}x_1^2 - \dot{x}_{1r} \tag{9.73}$$

we obtain

$$\dot{z}_1 = -c_1z_1 + (\tilde{k}_{10} - k_{10})10^8\tilde{x}_3x_1 + (\tilde{k}_{30} - k_{30})10^6\tilde{x}_3^{E_3/E_1}x_1^2 \tag{9.74}$$

Consider the modified lyapunov function

$$V_0 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma_1^2}(\tilde{k}_{10} - k_{10})^2 + \frac{1}{2\gamma_2^2}(\tilde{k}_{30} - k_{30})^2 \tag{9.75}$$

and its derivative

$$\begin{aligned}\dot{V}_0 &= z_1\dot{z}_1 + \frac{1}{\gamma_1^2}(\tilde{k}_{10} - k_{10})\dot{\tilde{k}}_{10} + \frac{1}{\gamma_2^2}(\tilde{k}_{30} - k_{30})\dot{\tilde{k}}_{30} \\ &= -c_1z_1^2 + (\tilde{k}_{10} - k_{10})\left[\frac{1}{\gamma_1^2}\dot{\tilde{k}}_{10} + 10^8\tilde{x}_3x_1z_1\right] \\ &\quad +(\tilde{k}_{30} - k_{30})\left[\frac{1}{\gamma_2^2}\dot{\tilde{k}}_{30} + 10^6\tilde{x}_3^{E_3/E_1}x_1^2z_1\right]\end{aligned} \tag{9.76}$$

We can now choose

$$\begin{aligned}\dot{\tilde{k}}_{10} &= -\gamma_1^2 10^8\tilde{x}_3x_1z_1 \\ \dot{\tilde{k}}_{30} &= -\gamma_2^2 10^6\tilde{x}_3^{E_3/E_1}x_1^2z_1\end{aligned} \tag{9.77}$$

to make the derivative of Lyapunov function independent of the unknown true parameters k_{10} and k_{30}

$$\dot{V}_0 = -c_1z_1^2$$

Step 2:

Note that we do not have the matching condition, *i.e.*, the uncertain parameter and the virtual input appear as a product. As in the previous case, (9.59) can be written as (substituting for $\tilde{x}_{3r}$) after augmenting it with $\frac{1}{\gamma_3^2}(\tilde{k}_{10} - k_{10})^2$

$$\begin{aligned}\dot{V}_1 &= -c_1z_1^2 + z_2\left[-u_1x_2 + \frac{k_{10}}{\tilde{k}_{10}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r}) - \dot{x}_{2r}\right] + \frac{1}{\gamma_3^2}(\tilde{k}_{10} - k_{10})\dot{\tilde{k}}_{10} \\ &\quad +z_2z_3(k_{10}x_1 - k_{20}x_2)\end{aligned} \tag{9.78}$$

Choosing

$$\begin{aligned}\dot{\tilde{k}}_{10} &= \dot{\tilde{k}}_{20} \\ &= \frac{\gamma_3^2}{\tilde{k}_{10}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2\end{aligned} \tag{9.79}$$

we get

$$\dot{V}_1 = -c_1z_1^2 - c_2z_2^2 + z_2z_3(k_{10}x_1 - k_{20}x_2) \tag{9.80}$$

Note that these are the parameter update laws for the uncertain parameters in the first virtual input $\tilde{x}_{3r}$ and (9.77) are the update laws for the unknown parameters in u_1.

Step 3:

Augmenting the Lyapunov function V_2 with $\frac{1}{\gamma_4^2}(\tilde{k}_{10} - k_{10})^2 + \frac{1}{\gamma_5^2}(\tilde{k}_{30} - k_{30})^2$, we get

$$\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_2 z_3 (\tilde{k}_{10} - k_{10})(x_1 - x_2) \\
&\quad + \frac{z_3}{\rho C_\rho}[(\tilde{k}_{10} - k_{10}) 10^8 \tilde{x}_3 x_1 d_1 + (\tilde{k}_{20} - k_{20}) 10^8 \tilde{x}_3 x_2 d_2 \\
&\quad + (\tilde{k}_{30} - k_{30}) 10^6 \tilde{x}_3 x_1^2 d_3] + \frac{1}{\gamma_4^2}(\tilde{k}_{10} - k_{10})\dot{\tilde{k}}_{10} + \frac{1}{\gamma_5^2}(\tilde{k}_{30} - k_{30})\dot{\tilde{k}}_{30} \\
&= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + (\tilde{k}_{10} - k_{10})[z_2 z_3 (x_1 - x_2) + \frac{1}{\rho C_\rho}(10^8 \tilde{x}_3 x_1 d_1 + 10^8 \tilde{x}_3 x_2 d_2) z_3 \\
&\quad + \frac{1}{\gamma_4^2}\dot{\tilde{k}}_{10}] + (\tilde{k}_{30} - k_{30})[\frac{1}{\gamma_5^2}\dot{\tilde{k}}_{30} + \frac{1}{\rho C_\rho}(10^6 \tilde{x}_3 x_1^2 d_3 z_3)] \qquad (9.81)
\end{aligned}$$

The following choice of update laws would make the Lyapunov function independent of unknown parameters

$$\begin{aligned}
\dot{\tilde{k}}_{10} &= -\gamma_4^2 \left[z_2 z_3 (x_1 - x_2) + \frac{1}{\rho C_\rho}(10^8 \tilde{x}_3 x_1 d_1 + 10^8 \tilde{x}_3 x_2 d_2) z_3 \right] \\
\dot{\tilde{k}}_{30} &= -\gamma_5^2 \frac{1}{\rho C_\rho} \left[10^6 \tilde{x}_3 x_1^2 d_3 z_3 \right] \qquad (9.82)
\end{aligned}$$

These are the update laws for the virtual input x_{4r}. Including all the above update laws will result in

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 - c_4 z_3^2 \qquad (9.83)$$

Using Lasalle's theorem, it is easy to see that all the trajectories would go to $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$ asymptotically. This method of designing update laws can result in over parametrization as in this case. We have two uncertain parameters and five update laws. This problem can be avoided by designing tuning functions. At this point it is important to observe that this method does not guarantee the convergence of uncertain parameters to their true values. It can only guarantee asymptotic tracking of the outputs.

9.5.2 Tuning Function Design

In the previous section we tried to get rid of the unknown parameters at every step, but instead, for closed loop asymptotic tracking we can wait till the last step and then

cancel all the unknowns with just one update law for each uncertain parameter. The new update law will then be the sum of all the previous update laws for each uncertain parameter with the same adaptive gain. Each of the update laws that make up the final update law are called tuning functions. The new update laws can be shown to be

$$\begin{aligned}\dot{\hat{k}}_{10} &= -\gamma_1^2\left[10^8\tilde{x}_3x_1z_1+\frac{z_2}{\hat{k}_{10}}(-c_2z_2+u_1x_2+\dot{x}_{2r})+z_2z_3(x_1-x_2)\right.\\ &\quad\left.+\frac{1}{\rho C_\rho}(10^8\tilde{x}_3x_1d_1+10^8\tilde{x}_3x_2d_2)z_3\right] \qquad (9.84)\\ \dot{\hat{k}}_{30} &= -\gamma_2^2\left[10^6\tilde{x}_3^{E_3/E_1}x_1^2z_1+\frac{1}{\rho C_\rho}10^6\tilde{x}_3x_1^2d_3z_3\right] \qquad (9.85)\end{aligned}$$

9.6 Robust Adaptive Backstepping

In this section we assume that all the estimated constants are uncertain and then derive a controller that is robustly stable. Note that k_{i0} is a positive constant and E_i is a negative constant. We also assume that $E_1 = E_2$ and $k_{10} = k_{20}$. Further more suppose that the upper and the lower bounds, E_{01} and E_{02} respectively, of E_1 are known *i.e.*, E_{01} and E_{02} are such that $E_{02} \leq E_2 = E_1 \leq E_{01} < 0$. Since E_i are negative constants $k_i(x_3)$ are bounded for all $x_3 \geq 0$. Now considering the following new variable with an arbitrarily chosen $E > E_{01}$ (E can be considered as a tuning parameter) and a scaling factor k_n

$$\bar{x}_3 = k_n e^{E/x_3}$$

the controlled plant can be written as

$$\begin{aligned}\dot{x}_1 &= u_1(c_{A0}-x_1)-k_1(\bar{x}_3)x_1-k_3(\bar{x}_3)x_1^2 \qquad (9.86a)\\ \dot{x}_2 &= -u_1x_2+k_0(\bar{x}_3)\bar{x}_3(x_1-x_2) \qquad (9.86b)\\ \dot{\bar{x}}_3 &= -\bar{x}_3\frac{[ln(\bar{x}_3/k_n)]^2}{E}(u_1(\nu_0-\frac{E}{ln(\bar{x}_3/k_n)})+C_1(x_4-\frac{E}{ln(\bar{x}_3/k_n)})-\frac{1}{\rho C_p}f_{\bar{x}_3}) \qquad (9.86c)\\ \dot{x}_4 &= \frac{1}{m}u_2+\frac{C_2}{m}(\frac{E}{ln(\bar{x}_3/k_n)}-x_4) \qquad (9.86d)\\ & \qquad (9.86e)\end{aligned}$$

where

$$k_0(\bar{x}_3) = \frac{k_{10}}{k_n}e^{\frac{E_1-E}{x_3}} = \frac{k_{10}}{k_n}\left(\frac{\bar{x}_3}{k_n}\right)^{\frac{E_1-E}{E}}$$

Since $E_1 < E$, $k_0(\bar{x}_3)$ is bounded for all $x_3 \geq 0$. Please note the new definitions for the variables $k_1(\bar{x}_3)$, $k_3(\bar{x}_3)$, $k_0(\bar{x}_3)$ by comparing the above set of equations with the

original system (9.4).

In the following steps we propose a Lyapunov function(V_i) at each stage whose derivative is of the form

$$\dot{V}_i = -\alpha V_i + \beta$$

for a particular set of true and virtual inputs. α and β are some positive constants. Thus proving that the controller designed in this section is robustly stable. Also note that it is possible to make the control error as small as desired by an appropriate choice of tuning parameters.

Step 1:

Consider the following input u_1

$$u_1 = \frac{-c_1 z_1 + \dot{x}_{1r} + u_{R1}}{c_{A0} - x_1} \tag{9.87a}$$

$$u_{R1} = \begin{cases} -[\hat{k}_1(t)^T \bar{x}_1(t)]^2 z_1(t)/\epsilon_1 & if |\hat{k}_1 \bar{x}_1 z_1| \leq \epsilon_1 \\ -\hat{k}_1^T \bar{x}_1(t) sign(z_1) & if |\hat{k}_1 \bar{x}_1 z_1| > \epsilon_1 \end{cases} \tag{9.87b}$$

where $z_1 = x_1 - x_{1r}$, $\bar{x}_1^T = [x_1 \;\; x_1^2]$ and $\hat{k}_1 = [\hat{k}_{1max} \;\; \hat{k}_{3max}]$ and $\hat{k}_1$ is updated by

$$\dot{\hat{k}}_1(t) = \Gamma \bar{x}_1 |z_1| - \sigma_1 \hat{k}_1(t) \tag{9.88}$$

where $\Gamma = \Gamma^T$ is a positive definite matrix and $\sigma_1 > 0$ is scalar tuning parameter. $\hat{k}_{1max}$ and $\hat{k}_{3max}$ are the maximum estimated values of $k_1(\bar{x}_3)$, $k_3(\bar{x}_3)$ respectively.

From the assumption that $x_3 > 0$, we have

$$k_1(x_3)x_1 + k_3(x_3)x_1^2 \leq k_{1max}x_1 + k_{3max}x_1^2 = k_{1max}^T \bar{x}_1 \tag{9.89}$$

Now consider the following positive definite function $\bar{V}_0$

$$\bar{V}_0 = \frac{1}{2}z_1^2 + \frac{1}{2}(\hat{k}_1 - k_{1max})^T \Gamma^{-1} (\hat{k}_1 - k_{1max}) \tag{9.90}$$

The time derivative of $\bar{V}_0$ is then, if $|\hat{k}_1^T \bar{x}_1 z_1| > \epsilon_1$,

$$\begin{aligned} \dot{\bar{V}}_0 &= z_1 \dot{z}_1 + (\hat{k}_1 - k_{1max})^T \Gamma^{-1} \dot{\hat{k}}_1(t) \\ &\leq -c_1 z_1^2 - \sigma_1 \lambda_{min}[\Gamma^{-1}] \|\hat{k}_1 - k_{1max}\|^2 \\ &\quad + \sigma_1 \|\Gamma^{-1}\| \|\hat{k}_1 - k_{1max}\| \|k_{1max}\| \\ &\triangleq \bar{V}_{01} \end{aligned} \tag{9.91}$$

In the case where $|\hat{k}_1^T \bar{x}_1 z_1| \leq \epsilon_1$, we have

$$\dot{\bar{V}}_0 \leq \bar{V}_{01} + \epsilon_1$$

It is easy to see that this will ensure robust stability of the process *i.e.*, all the input, u_1, and the state, x_1, remain bounded. $\bar{V}_{01}$ can be rewritten as

$$\begin{aligned}
\dot{\bar{V}}_{01} &= -c_1 z_1^2 - \sigma_1 \lambda_{min}[\Gamma^{-1}] \|\hat{k}_1 - k_{1max}\|^2 \\
&\quad + \sigma_1 \|\Gamma^{-1}\| \|\hat{k}_1 - k_{1max}\| \|k_{1max}\| \\
&= -c_1 z_1^2 - (\sigma_1 \lambda_{min}[\Gamma^{-1}] - \zeta_1) \|\hat{k}_1 - k_{1max}\|^2 \\
&\quad - \zeta_1 \|\hat{k}_1 - k_{1max}\|^2 + \sigma_1 \|\Gamma^{-1}\| \|k_{1max}\| \|\hat{k}_1 - k_{1max}\|^2 \\
&\leq -c_1 z_1^2 - (\sigma_1 \lambda_{min}[\Gamma^{-1}] - \zeta_1) \|\hat{k}_1 - k_{1max}\|^2 \\
&\quad + \frac{(\sigma_1 \|\Gamma^{-1}\| \|k_{1max}\|)^2}{4\zeta_1}
\end{aligned} \tag{9.92}$$

By choosing ζ_1 as

$$\zeta_1 = \frac{1}{2} \sigma_1 \lambda_{min}[\Gamma^{-1}]$$

we have

$$\dot{\bar{V}}_0 \leq -\alpha \bar{V}_0 + \beta \tag{9.93}$$

where

$$\begin{aligned}
\alpha &= \min \left(2c_1, \frac{\sigma_1 \lambda_{min}[\Gamma^{-1}]}{\lambda_{max}[\Gamma^{-1}]} \right) \\
\beta &= \frac{\sigma_1 (\|\Gamma^{-1}\| \|k_{1max}\|)^2}{2\lambda_{min}[\Gamma^{-1}]} + \epsilon_1
\end{aligned} \tag{9.94}$$

From the above inequality and the fact that $z_1^2 \leq 2\bar{V}_0$, it follows that

$$\lim_{t \to \infty} |z_1|^2 \leq 2\frac{\beta}{\alpha}$$

Thus, for any δ such that $\delta^2 \geq \beta/\alpha$ we can achieve

$$\lim_{t \to \infty} |z_1| \leq \delta \tag{9.95}$$

It is also clear that appropriate choice of Γ and ϵ_1 ensures the inequality. For instance, it is sufficient to set

$$\begin{aligned}
\Gamma &= \gamma I_2 \\
\gamma &> \frac{2\sigma_1 \|k_{max}\|^2}{\alpha \delta^2} \\
\epsilon_2 &< \frac{\alpha \delta^2}{4}
\end{aligned} \tag{9.96}$$

where $\alpha = \min(2c_1,\ \delta_1)$. It can similarly be proved that z_2, z_3 and z_4 can be reduced to a desired level by appropriately choosing the tuning parameters.

Step 2:

Define $z_2 = x_2 - x_{2r}$. It then follows that

$$\dot{z}_2 = -u_1 x_2 + k_0(\bar{x}_3)\bar{x}_3(x_1 - x_2) - \dot{x}_{2r} \tag{9.97}$$

Choosing

$$\bar{x}_{3r} = \frac{-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}}{\hat{k}_{0max}(x_1 - x_2)} + u_{R2} \tag{9.98}$$

where $\hat{k}_{0max}$ is the estimated maximum value of $k_0(\bar{x}_3)$ and u_{R2} is a robust control term. γ_2, c_2 and σ_2 are positive tuning parameters. The update law for $\hat{k}_{0max}$ and robust control signal are designed as follows

$$\dot{\hat{k}}_{0max} = \gamma_2 \frac{|-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}|}{\hat{k}_{0max}}|z_2| - \sigma_2 \hat{k}_{0max} \tag{9.99}$$

with the initial condition $\hat{k}_{max}(0) > 0$

$$u_{R2} = \begin{cases} 0 & \text{if } (-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}) z_2 \geq 0 \\ -\frac{2(-c_2 z_2 + u_1 x_2 + \dot{x}_{2r})^2}{\epsilon_2 k_{0m}(x_3)(x_1 - x_2)} & \text{if } -\epsilon_2 \leq (-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}) z_2 < 0 \\ \frac{2(-c_2 z_2 + u_1 x_2 + \dot{x}_{2r})}{k_{0m}(x_3)(x_1 - x_2)} & \text{if } (-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}) z_2 < -\epsilon_2 \end{cases} \tag{9.100}$$

for $\epsilon_2 > 0$. Where

$$k_{0m}(x_3) = \frac{k_{0m}}{k_n} e^{\frac{E_2 - E}{x_3}} < k_0(\bar{x}_3)$$

k_{0m} is a lower bound of k_{10}. Considering a positive definite function

$$\bar{V}_1 = \frac{1}{2} z_2^2 + \frac{1}{2\gamma_2}(\hat{k}_{0max} - k_{0max})^2 \tag{9.101}$$

we have

$$\begin{aligned} \dot{\bar{V}}_1 &= (-u_1 x_2 - \dot{x}_{2r}) z_2 + k_0(\bar{x}_3) z_3 (x_1 - x_2) z_2 + k_0(\bar{x}_3) u_{R2}(x_1 - x_2) z_2 \\ &\quad + \frac{k_0(\bar{x}_3)}{\hat{k}_{0max}}(-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}) z_2 + \frac{1}{\gamma_1}(\hat{k}_{0max} - k_{0max})\dot{\hat{k}}_{0max} \\ &\leq -c_2 z_2^2 + k_0(\bar{x}_3)(x_1 - x_2) z_2 z_3 + k_0(\bar{x}_3) u_{R2}(x_1 - x_2) z_2 \\ &\quad + \frac{k_0(\bar{x}_3)}{\hat{k}_{0max}}(-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}) z_2 - (-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}) z_2 \\ &\quad + |-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}||z_2| - \frac{k_{0max}}{\hat{k}_{0max}}|-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}||z_2| \\ &\quad - \frac{\sigma_2}{\gamma_2}(\hat{k}_{0max} - k_{0max})^2 + \frac{\sigma_2}{\gamma_2}|\hat{k}_{0max} - k_{0max}| k_{0max} \end{aligned} \tag{9.102}$$

If $(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 \geq 0$ then we have

$$\begin{aligned}\dot{\bar{V}}_1 &\leq -c_2z_2^2 + k_0(\bar{x}_3)(x_1 - x_2)z_2z_3 - \frac{\sigma_2}{\gamma_2}(\hat{k}_{0max} - k_{0max})^2 \\ &\quad + \frac{\sigma_2}{\gamma_2}|\hat{k}_{0max} - k_{0max}|k_{0max} \\ &\triangleq \bar{V}_{11} + k_0(\bar{x}_3)(x_1 - x_2)z_2z_3\end{aligned} \tag{9.103}$$

If $(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 < -\epsilon_2$ then

$$\begin{aligned}\dot{\bar{V}}_1 &\leq \bar{V}_{11} + \frac{k_0(\bar{x}_3)}{k_{0m}(x_3)}2(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 \\ &\quad + 2|-c_2z_2 + u_1x_2 + \dot{x}_{2r}||z_2| + k_0(\bar{x}_3)(x_1 - x_2)z_2z_3 \\ &\leq \bar{V}_{11} + k_0(\bar{x}_3)(x_1 - x_2)z_2z_3\end{aligned} \tag{9.104}$$

In the case where $\epsilon_2 \leq (-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 < 0$ we have

$$\dot{\bar{V}}_1 \leq \bar{V}_{11} + k_0(\bar{x}_3)(x_1 - x_2)z_2z_3 + 2\epsilon_2 \tag{9.105}$$

Step 3:
Defining

$$z_3 = \bar{x}_3 - \bar{x}_{3r}$$

we obtain

$$\begin{aligned}\frac{1}{C_1}\dot{z}_3 &= -\bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}x_4 - \bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}\left[\frac{1}{C_1}u_1(\nu_0 - \frac{E}{ln(\bar{x}_3/k_N)})\right. \\ &\quad \left. - \frac{E}{ln(\bar{x}_3/k_n)} - \frac{1}{\rho C_1C_p}f_x\right] - \frac{1}{C_1}\dot{x}_{3r}\end{aligned} \tag{9.106}$$

where

$$\begin{aligned}\frac{1}{\rho C_1C_p}f_x &= \frac{1}{\rho C_1C_p}(k_1(x_3)x_1d_1 + k_2(x_3)x_2d_2 + k_3(x_3)x_1^2d_3) \\ &\leq k_{d_1max}x_1 + k_{d_2max}x_2 + k_{d_3max}x_1^2 \\ &= k_{dmax}^T\bar{x}_f \qquad \bar{x}_f = [x_1 \;\; x_2 \;\; x_1^2]^T\end{aligned} \tag{9.107}$$

Choosing x_{4r} as

$$\begin{aligned}x_{4r} &= \frac{E}{ln(\bar{x}_3/k_n)} + \frac{c_3E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}z_3 \\ &\quad + u_{R3} + u_{R4} - \frac{E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}u_{R5} - \frac{E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}\end{aligned} \tag{9.108}$$

where

$$u_{R3} = \hat{\beta}_1 \left[u_1(\nu_0 - \frac{E}{ln(\bar{x}_3/k_n)}) + \frac{E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}\dot{x}_{3r} \right] \tag{9.109}$$

$$\dot{\hat{\beta}}_1 = \gamma_3 \frac{-\bar{x}_3(ln(\bar{x}_3/k_n))^2}{E} \left[u_1(\nu_0 - \frac{E}{ln(\bar{x}_3/k_n)}) + \frac{E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}\dot{x}_{3r} \right] z_3 - \sigma_3\hat{\beta}_1 \tag{9.110}$$

$$\gamma_3 > 0 \quad \sigma_3 > 0$$

$$u_{R4} = \begin{cases} -\left[\hat{k}_d^T\bar{x}_f|\frac{\bar{x}_3 ln(\bar{x}_3/k_n)^2}{E}|\right]^2 z_3/\epsilon_4 & \text{if } |\hat{k}_d^T\bar{x}_f|\frac{\bar{x}_3 ln(\bar{x}_3/k_n)^2}{E}|z_3| \le \epsilon_4 \\ -\hat{k}_d^T\bar{x}_f|\frac{\bar{x}_3 ln(\bar{x}_3/k_n)^2}{E}|sign(z_3) & \text{if } |\hat{k}_d^T\bar{x}_f|\frac{\bar{x}_3 ln(\bar{x}_3/k_n)^2}{E}|z_3| > \epsilon_4 \end{cases} \tag{9.111}$$

$$\dot{\hat{k}}_d = \Gamma_d\bar{x}_f|\frac{\bar{x}_3 ln(\bar{x}_3/k_n)^2}{E}||z_3| - \sigma_d\hat{k}_d \tag{9.112}$$

$$\Gamma_d = \Gamma_d^T > 0, \quad \sigma_d > 0$$

$$u_{R5} = \begin{cases} -\left[\hat{k}_5|z_2(x_1 - x_2)|\right]^2 z_3/\epsilon_5 & \text{if } |z_2(x_1 - x_2)z_3| \le \epsilon_5 \\ -\hat{k}_4|z_2(x_1 - x_2)|sign(z_3) & \text{if } |z_2(x_1 - x_2)z_3| > \epsilon_5 \end{cases} \tag{9.113}$$

$$\dot{\hat{k}}_5 = \gamma_5|z_2(x_1 - x_2)||z_3| - \sigma_5\hat{k}_5 \tag{9.114}$$

$$\sigma_5 > 0 \quad \gamma_5 > 0$$

By considering the following positive definite function

$$\bar{V}_2 = \bar{V}_1 + \frac{1}{2C_1}z_3^2 + \frac{1}{2\gamma_3}(\hat{\beta}_1 - \frac{1}{C_1})^2 + \frac{1}{2}(\hat{k}_d - k_{dmax})^T\gamma_d^{-1}(\hat{k}_d - k_{dmax}) + \frac{1}{2\gamma_5}(\hat{k}_5 - k_{0max})^2 \tag{9.115}$$

we have

$$\begin{aligned} \dot{\bar{V}}_2 \le\ & \dot{\bar{V}}_{11} + 2\epsilon_f - \bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}z_3z_4 - c_3z_3^2 + k_{0max}|z_2(x_1 - x_2)||z_3| \\ & + \bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}(\hat{\beta}_1 - \frac{1}{C_1})\left[u_1(\nu_0 - \frac{E}{ln(\bar{x}_3/k_n)}) + \frac{E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}\dot{x}_{3r} \right] z_3 \\ & + k_{dmax}^T\bar{x}_f|\bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}||z_3|u_{R4} + u_{R5} \\ & - \bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}(\hat{\beta}_1 - \frac{1}{C_1})\left[u_1(\nu_0 - \frac{E}{ln(\bar{x}_3/k_n)}) + \frac{E}{\bar{x}_3(ln(\bar{x}_3/k_n))^2}\dot{x}_{3r} \right] z_3 \\ & - \frac{\sigma_3}{\gamma_3}(\hat{\beta}_1 - \frac{1}{C_1})^2 + \frac{\sigma_3}{\gamma_3}|\hat{\beta}_1 - \frac{1}{C_1}|\frac{1}{C_1} + (\hat{k}_d - k_{dmax})^T\bar{x}_f|\bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}||z_3| \\ & - \sigma_d\lambda_{min}[\Gamma^{-1}]\|\hat{k}_d - k_{dmax}\|^2 + \sigma_d\|\Gamma_d^{-1}\|\|\hat{k}_d - k_{dmax}\|\|k_{dmax}\| \\ & + (\hat{k}_5 - k_{0max})|z_2(x_1 - x_2)||z_3| - \frac{\sigma_5}{\gamma_5}(\hat{k}_5 - k_{0max})^2 + \frac{\sigma_5}{\gamma_5}|\hat{k}_5 - k_{0max}|k_{0max} \end{aligned} \tag{9.116}$$

Finally we obtain

$$\begin{aligned}\dot{\bar{V}}_2 &\leq \bar{V}_{11} + 2\epsilon_f - \bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}z_3z_4 - c_3z_3^2 - \frac{\sigma_3}{\gamma_3}(\hat{\beta}_1 - \frac{1}{C_1})^2 + \frac{\sigma_3}{\gamma_3}|\hat{\beta}_1 - \frac{1}{C_1}|\frac{1}{C_1} \\ &\quad -\sigma_d\lambda_{min}[\Gamma^{-1}]\|\hat{k}_d - k_{dmax}\|^2 + \sigma_d\|\Gamma^{-1}\|\|\hat{k}_d - k_{dmax}\|\|k_{dmax}\| \\ &\quad -\frac{\sigma_5}{\gamma_5}(\hat{k}_5 - k_{0max})^2 + \frac{\sigma_5}{\gamma_5}|\hat{k}_5 - k_{0max}|k_{0max} + \epsilon_4 + \epsilon_5 \\ &= \bar{V}_{11} + 2\epsilon_f + \bar{V}_{21} + \epsilon_4 + \epsilon_5 - \bar{x}_3\frac{(ln(\bar{x}_3))^2}{E}z_3z_4\end{aligned}$$

where

$$\begin{aligned}\bar{V}_{21} &= -c_3z_3^2 - \frac{\sigma_3}{\gamma_3}(\hat{\beta}_1 - \frac{1}{C_1})^2 + \frac{\sigma_3}{\gamma_3}|\hat{\beta}_1 - \frac{1}{C_1}|\frac{1}{C_1} - \sigma_d\lambda_{min}[\Gamma^{-1}]\|\hat{k}_d - k_{dmax}\|^2 \quad (9.117) \\ &\quad +\sigma_d\|\Gamma^{-1}\|\|\hat{k}_d - k_{dmax}\|\|k_{dmax}\| - \frac{\sigma_5}{\gamma_5}(\hat{k}_5 - k_{0max})^2 + \frac{\sigma_5}{\gamma_5}|\hat{k}_5 - k_{0max}|k_{0max}\end{aligned}$$

Step 4:

The fourth error system is

$$\begin{aligned}z_4 &= x_4 - x_{4r} \quad (9.118) \\ m\dot{z}_4 &= u_2 + C_2(\frac{E}{ln(\bar{x}_3/k_n)} - x_4) - m\dot{x}_{4r}\end{aligned}$$

We design u_2 by

$$u_2 = -c_2z_4 + \bar{x}_3\frac{(ln(\bar{x}_3/k_n))^2}{E}z_3 - \hat{C}_2(\frac{E}{ln(\bar{x}_3/k_n)} - x_4) + \hat{m}\dot{x}_{4r} \quad (9.119)$$

where

$$\begin{aligned}\dot{\hat{C}}_2 &= \gamma_6(\frac{E}{ln(\bar{x}_3/k_n)} - x_4)z_4 - \sigma_6\hat{C}_2 \quad (9.120) \\ \dot{\hat{m}} &= -\gamma_7\dot{x}_{4r}z_4 - \sigma_4\hat{m} \quad (9.121)\end{aligned}$$

$$\gamma_6 > 0 \quad \sigma_6 > 0 \quad \gamma_7 > 0 \quad \sigma_7 > 0$$

Setting a final positive definite function by

$$\bar{V}_3 = \bar{V}_2 + \frac{1}{2}mz_4^2 + \frac{1}{2\gamma_6}(\hat{C}_2 - C_2)^2 + \frac{1}{2\gamma_7}(\hat{m} - m)^2 \quad (9.122)$$

we have

$$\begin{aligned}\dot{\bar{V}}_3 &\leq \bar{V}_{11} + \bar{V}_{21} + 2\epsilon_f + \epsilon_4 + \epsilon_5 - c_4z_4^2 - \frac{\sigma_6}{\gamma_6}(\hat{C}_2 - C_2)^2 + \frac{\sigma_6}{\gamma_6}|\hat{C}_2 - C_2|C_2 \quad (9.123) \\ &\quad -\frac{\sigma_7}{\gamma_7}(\hat{m} - m)^2 + \frac{\sigma_7}{\gamma_7}|\hat{m} - m|m\end{aligned}$$

From the the Eqns (9.105), (9.117), (9.122), one can easily show that all the signals in the control system are bounded so that the process is robustly stable. A method similar to the one in Step 1 can be used to prove that the errors converge to a value less than the specified error bound. The key to achieving desired error bounds lies in choosing appropriate values of γ_i, σ_i and ϵ_i.

9.7 Simulations and Results

The particular CSTR model used in this project as well as in many other papers is highly nonlinear and very difficult to control. In (Chen *et al.* 1995) an optimum steady state of the yield Φ of product B with respect to the design variables was obtained[2]. They found the following optimal operating point

$$\begin{array}{rclrcl} c_{A0}|_s & = & 5.10 \text{ mol/l}, & c_A|_s & = & 2.14 \text{ mol/l}, \\ \nu_0|_s & = & 104.9^o \text{ C}, & c_B|_s & = & 1.09 \text{ mol/l}, \\ \frac{\dot{V}}{V_R}|_s & = & 14.9\ h^{-1}, & \nu|_s & = & 114.2^o \text{ C}, \\ \dot{Q}_K|_s & = & -1113.5 \text{ kJ/h}, & \nu_K|_s & = & 112.9^o \text{ C}. \end{array}$$

However under real operating conditions, it is quite often required to change the operating point. Moreover, the presence of disturbances makes it very important to regulate the outputs. The open loop behaviour of the CSTR model under increments of +18% and -18% in the steady state value of ν_0 is shown in fig.9.2. The controller performance for $c_1 = 1$, $c_2 = 2$, $c_3 = 0.1$, $c_4 = 1$ is shown in fig.9.3. It was found that the transient performance of the controller can be improved by an appropriate choice of c_i's, but it is difficult to define and find what is appropriate. The above choice of c_i's was found to give reasonable performance (it was obtained by trial and error). The performance of the controller under disturbances is shown in fig.9.4. A +18% increase in ν_0 was introduced at $t = 500sec$. The results obtained by avoiding the cancellation of useful nonlinearities are shown in fig.9.5. The useful nonlinearities in this problem are not really strong and therefore, we don't see much difference between this controller and the original one. The controller performance in the absence of the derivatives is shown in fig.9.6. It can be observed that the inputs are much smaller in magnitude in this case. Fig.9.7 shows the performance of a nonadaptive backstepping controller which uses estimated parameters instead of the true parameters. The following estimated

[2] Note that the figure 1 also gives the same optimum c_A and c_B

Table 9.2: Various tuning parameters used in simulations

Figure	c_1	c_2	c_3	c_4	γ_1	γ_2	γ_3	γ_4	γ_5
Figures 9.3,9.4,9.5	1	2	0.1	1	-	-	-	-	-
Figures 9.6,9.7	1	1	0.001	1	-	-	-	-	-
Figures 9.8,9.9(a),9.9(b),9.10	1	1	0.001	1	10	10	5	0.1	0.05
Figures 9.11,9.12	0.001	0.1	0.001	0.01	0.1	0.1	-	-	-

values of the uncertain parameters are used.

$$k_{10} = 4 \times 10^8 \quad k_{20} = 4 \times 10^8 \quad k_{30} = 2.8 \times 10^6$$

Clearly, if the estimated values are not the true values a nonadaptive backstepping controller can not track. The performance of the adaptive controller with update laws for the uncertain parameters is shown in fig.9.8. The initial values of the estimated parameters are same as above. The uncertain parameters from the five update laws obtained from the original method are shown in figs.9.9(a),9.9(b),9.10. The controller response from the modified method along with the tuning function update laws are shown in figs.9.11,9.12. A disturbance of magnitude +18 % of steady state value of v_0 is introduced at $t = 17500sec$. Observe that both inputs in this figure are with in achievable levels. It is made possible by an appropriate choice of tuning parameters (c_i's). But then the time taken to reach the set point is really large. It is a trade off between the performance of the controller and the input energy spent. Finally, the results obtained by the robust adaptive controller are shown in fig.9.13. The Table 9.3 shows the various tuning parameters used in simulations.

[2]diag(x) stands for a diagonal matrix with elements of vector x along the diagonal
[3]The zero in the brackets denotes the initial value of this parameter

Table 9.3: Tuning parameters used in the Robust adaptive controller

Tuning Parameter	Value	Tuning Parameter	Value
Γ	$[1\ 0;0\ 1]$	σ_1	$1e-8$
γ_2	$1e-6$	σ_2	$1e-8$
γ_3	0.1	σ_3	0.01
Γ_d	$\text{diag}(1e-2, 1e-3, 1e-2)^3$	σ_d	$1e-4$
γ_5	0.1	σ_5	$1e-6$
γ_6	0.001	σ_6	0.01
γ_7	0.001	σ_7	0.01
c_1	6	c_2	30
c_3	0.1	c_4	15
E	-9500	k_{10}	$3.575e8$
k_n	$3.575e5$	$\hat{m}(0)$	0
e_1	0.01	e_2	0.1
e_4	0.1	e_5	0.004
$\hat{k}(0)^4$	$[0\ ;\ 0]$	$\hat{k}_{0max}(0)$	$1e2e^{(E1-E)/387.2}$
$\hat{\beta}(0)$	$0.1/C_1$	$\hat{k}_d(0)$	$[0.01\ \ 0.01\ \ 0.01]^T$
$\hat{k}_5(0)$	$10e^{(E1-E)/387.2}$	$\hat{C}_2(0)$	0
E_{02}	$1.1E$		

9.8 Conclusions

We have applied the technique of backstepping to design a nonlinear controller for a benchmark CSTR and studied its performance. We have also designed an adaptive backstepping controller that can track signals in presence of large uncertainties in the estimated plant. Tuning functions are designed to reduce the number of update laws. Finally a robust nonlinear adaptive controller is derived using Lyapunov stability theory. Though, the backstepping approach is simple and efficient, it has a few drawbacks. A backstepping controller can not guarantee asymptotic tracking in presence of state or output noise and/or input constraints. We think input constraints can be incorporated into this scheme. This forms part of our future work

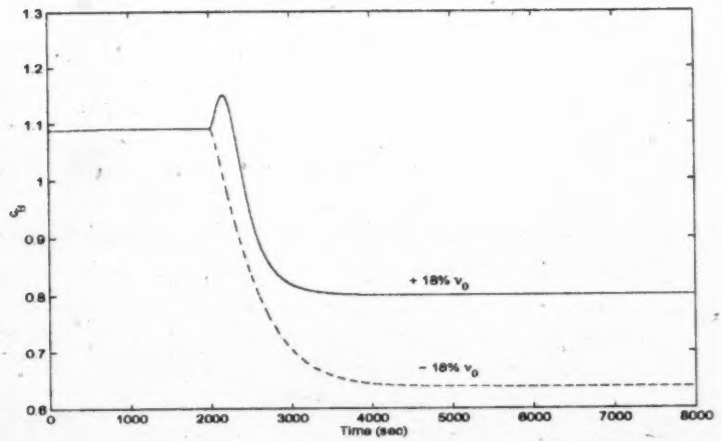


Figure 9.2: Open loop behaviour of the yield of product B under changes in temperature ν_0

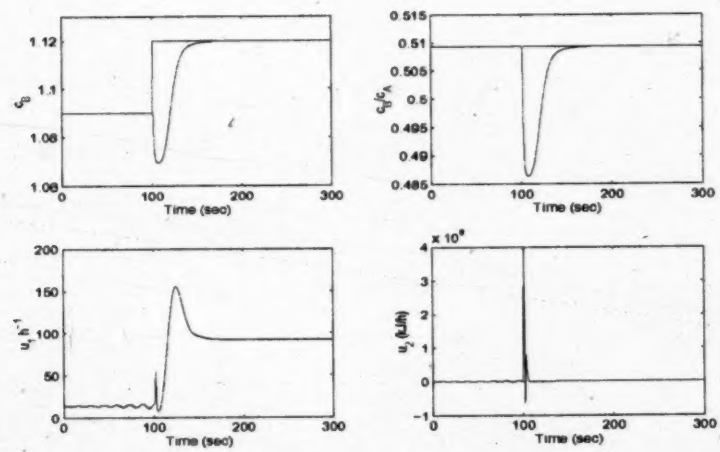


Figure 9.3: Setpoint tracking and the input magnitudes are shown in this figure

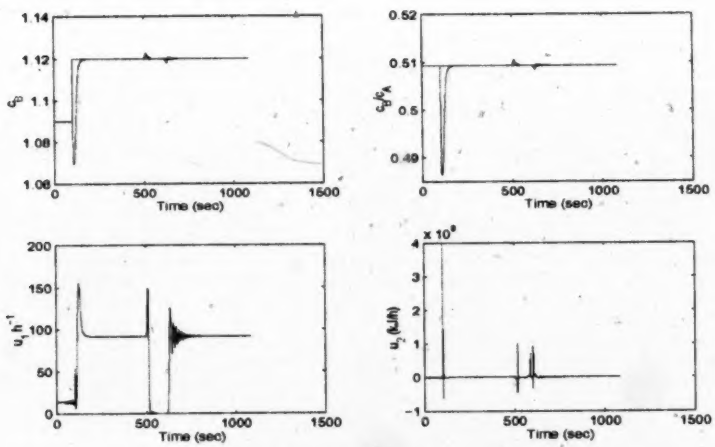


Figure 9.4: Disturbance rejection

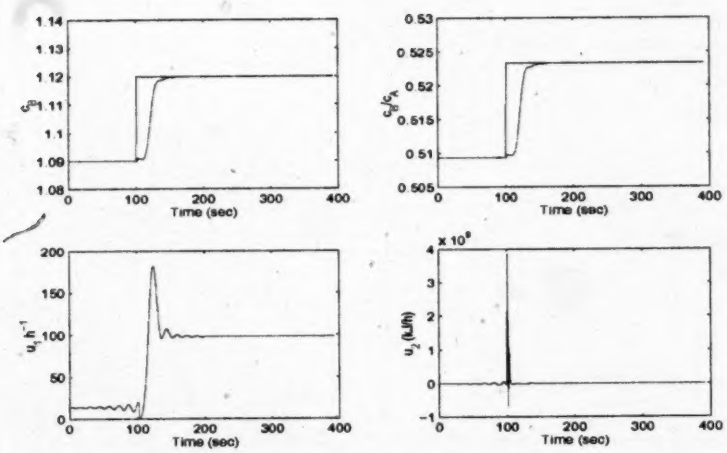


Figure 9.5: Controller without cancelling useful nonlinearities

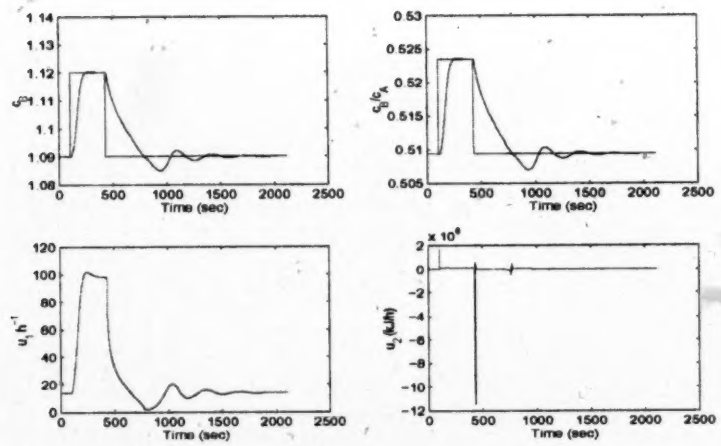


Figure 9.6: Tracking in the absence of derivatives

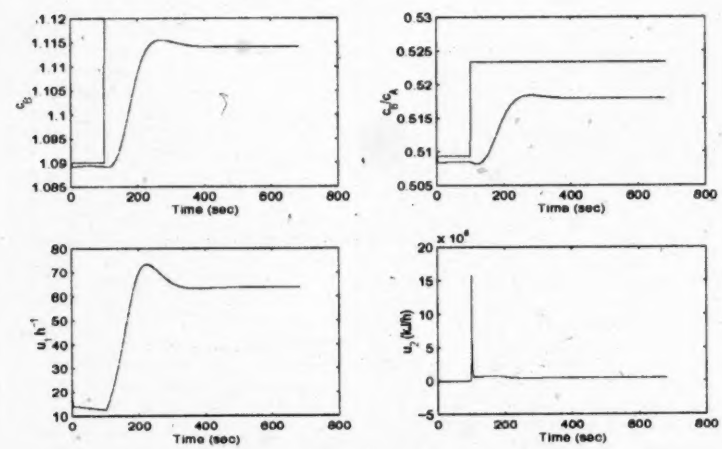


Figure 9.7: Tracking with a nonadaptive controller in presence of uncertain parameters

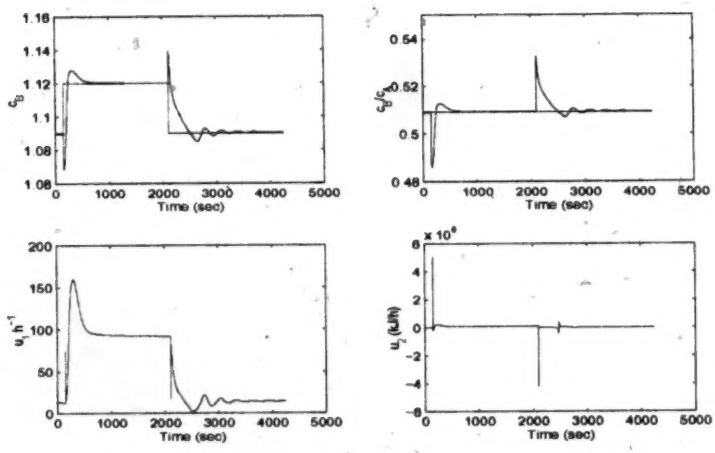
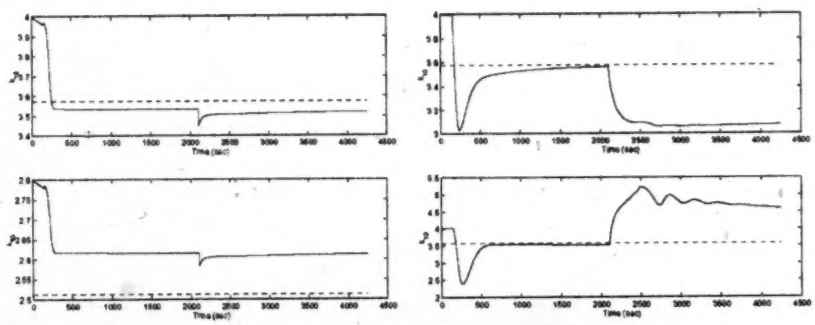


Figure 9.8: Adaptive backstepping



(a) The uncertain parameters from the first two update laws

(b) The uncertain parameters from the third and fourth update laws

Figure 9.9: Adaptation profiles of uncertain parameters

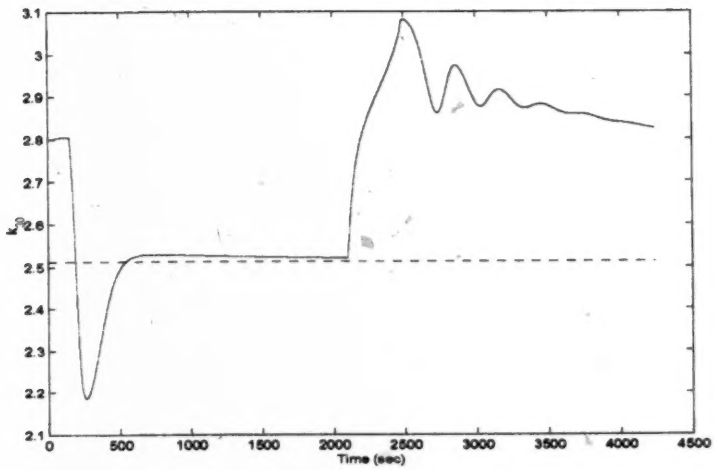


Figure 9.10: The uncertain parameters from the fifth update law

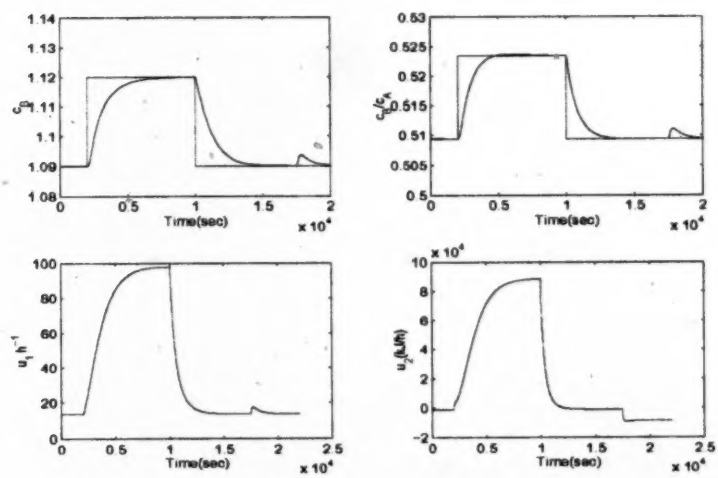


Figure 9.11: Step tracking using the modified controller

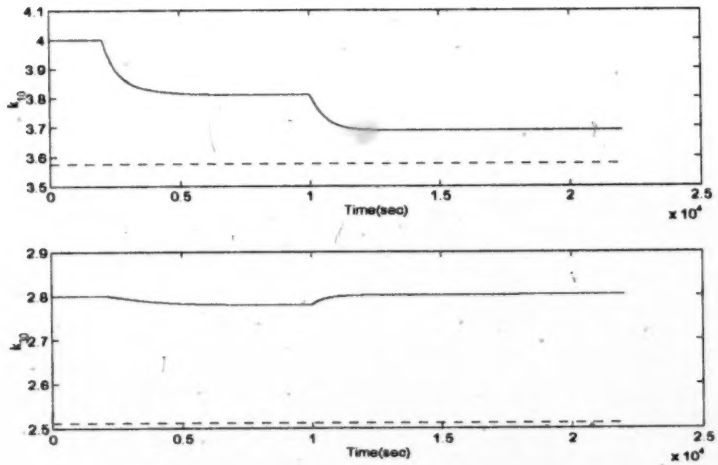


Figure 9.12: Update laws in (9.84), (9.85)

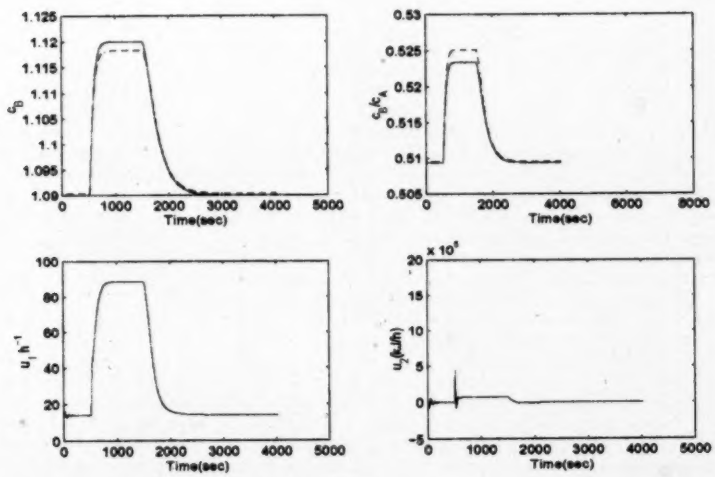


Figure 9.13: Response of the Robust adaptive controller - Thick line is the set point and the dashed line is the CSTR output

Part IV

Learning Theory and System Identification

10

Old wine in a new bottle : A learning theoretic approach to system identification

System identification involves using finite input-output data sets to identify the best model from a class of model sets. The quality of the identified model depends on the data collected. However, it is of interest both to theoreticians and practicians to know if the identified models do converge to the "true" best model as the data length tends to infinity and if they converge fast enough with respect to the data length. Learning theory is an useful tool that can be used to answer some of these questions in the context of system identification. In the current chapter, some background necessary to understand learning theory is presented. We modify the results in Vidyasagar (2001) to account for time varying nature of the identification objective function and also generalize a result providing upper bounds on the pollard dimension of model sets.[1]

10.1 Introduction

Traditional system identification techniques provide asymptotic properties of the estimated models. Such asymptotic properties of the estimated models are well documented in Ljung (1999). Even though asymptotic properties are useful only for

[1] An extended version with more results will be submitted to IFAC SysID Conference, 2003

large data sets, they provide useful insights into accuracy of the identified model. On the other hand, statistical learning theory is devoted to the derivation of finite time properties of measurable functions. If system identification is to be used for designing robust controllers, there is a need to develop a sound theory of finite sample properties of the estimated models. Therefore, a reformulation of system identification problems in terms of statistical learning theory is useful.

In this chapter, we provide some of the background material useful in understanding learning theory and present a few finite sample properties for prediction error models.

10.2 Background

In this section some of the concepts needed to understand learning theoretic approaches are presented. The main purpose of this section is to collect all the known definitions and results from measure theoretic probability that are useful in understanding learning theory. Most of the material in this section can be found in Vidyasagar (1997). For more detailed information on the measure theoretic concepts of probability readers are referred to Resnick (2001) and for information learning theory readers are referred to Vapnik (1998). Please note that not all the definitions and concepts presented in this section are necessary to understand what is presented in this chapter. Nevertheless, the following theorems and definitions are useful in further studying learning theory and its relation to system identification.

Definition 10.2.1 (Pseudometric Space) *Suppose X is a set. A function $\rho : X \times X \to \mathbb{R}^+$ is said to be* ***pseudometric*** *if*

(i) $\rho(x,x) = 0, \quad \forall\, x \in X$

(ii) $\rho(x,y) = \rho(y,x), \quad \forall\, x,y \in X$, *and*

(iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z), \quad \forall\, x,y,z \in X$

If, in addition, ρ satisfies

(iv) $\rho(x,y) = 0$ *if and only if* $x = y$

then ρ is said to be a metric

Suppose (X,ρ) is a pseudometric space, and that $x \in X$, $\epsilon > 0$. Then we denote

$$\begin{aligned} \mathcal{B}(\epsilon,x,\rho) &:= \{y \in X : \rho(x,y) < \epsilon\} \quad \text{and} \\ \bar{\mathcal{B}}(\epsilon,x,\rho) &:= \{y \in X : \rho(x,y) \leq \epsilon.\} \end{aligned} \tag{10.1}$$

$\mathcal{B}(\epsilon,x,\rho)$ and $\bar{\mathcal{B}}(\epsilon,x,\rho)$ are called open and closed balls of radius ϵ, centered at ρ.

Definition 10.2.2 (ϵ-covering number) *Let (X, ρ) be a pseudometric space and $S \subseteq X$. Given $\epsilon > 0$, a set $\{a_1, \cdots, a_n\} \subseteq S$ is said to be an ϵ-cover of S if*

$$\bigcup_{i=1}^{n} \bar{B}(\epsilon, a_i, \rho) \supseteq S. \tag{10.2}$$

The ϵ-covering number of S is defined as the smallest number n such that S has an ϵ cover of cardinality n and is denoted by $N(\epsilon, \rho, S)$. Similarly, external ϵ-cover is defined with respect to a set $\{b_1, \cdots, b_m\} \subseteq X$ if

$$\bigcup_{i=1}^{m} \bar{B}(\epsilon, b_i, \rho) \supseteq S. \tag{10.3}$$

External ϵ-cover can similarly be defined as the largest number m such that the set S has an ϵ-cover of cardinality m and is denoted by $L(\epsilon, S, \rho)$.

Remark 10.2.1 It is easy to observe that every internal ϵ-cover is also an external ϵ-cover. According to Vidyasagar (1997), the above definitions of covers and covering numbers are not standard. Some of the authors define the (external) ϵ-covering number as the smallest number of open balls of radius ϵ, as opposed to the smallest number of closed balls of radius ϵ.

Definition 10.2.3 (Compact Set) *Suppose (X, ρ) is a pseudometric space and that $S \subseteq X$. Then the set S is said be compact if every open cover of S has a finite subcover.*

Definition 10.2.4 (Totally Bounded Set) *A set is said to be totally bounded if*

$$N(\epsilon, S, \rho) < \infty \qquad \forall\, \epsilon > 0 \tag{10.4}$$

In other words, if the set S has a finite ϵ-covering number for all positive ϵ, the set S is said to be totally bounded.

Lemma 10.2.1 *Suppose that (X, ρ) is a pseudometric space, and that $S \subseteq X$. Then S is compact if and only if it is totally bounded and closed.*

Definition 10.2.5 (Sigma Algebra) *Suppose X is a set. A nonempty collection of subsets of X, $\mathcal{S}$, is said to be a σ-algebra if it satisfies the following conditions :*

(i) $\mathcal{S}$ is closed under complementation i.e., if $A \in \mathcal{S}$ then $A^c \in \mathcal{S}$

(ii) $\mathcal{S}$ is closed under countable union i.e., if $A_i \in \mathcal{S}$ for $i \in \mathbb{Z}^+$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$

Remark 10.2.2 In particular, if (X, ρ) is a pseudometric space, then the smallest σ-algebra of subsets of X that contains every closed subset of X is called the **Borel σ-algebra** of (X, ρ). It can easily be proved that the Borel σ-algebra also contains every open subset of X.

Definition 10.2.6 (Measurable space) *If X is a set and $\mathcal{S}$ is a collection of subsets of X, then the pair $(X, \mathcal{S})$ is called a* ***measurable space***

Definition 10.2.7 (Measurable function) *Suppose $(X, \mathcal{S}$ and $(Y, \mathcal{T})$ are measurable spaces, then a function $f : X \to Y$ is said to be measurable if for any open set $T \in \mathcal{T}$, $f^{-1}(T) \in \mathcal{S}$.*

From the above definitions, it is easy to see that probability is a measure on an event space with the following additional properties :

(i) $P(\phi) = 0$ and $P(X) = 1$ where ϕ is the null set

(ii) $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$.

Suppose that $(\Omega, \mathcal{T}, Q)$ is a probability space and $\{f_m\}_{m \geq 1}$ is a stochastic process on $(\Omega, \mathcal{T}, Q)$. Thus, for each m, $f_m : \Omega \to \mathbb{R}$ is measurable. Let $g : \Omega \to \mathbb{R}$ be another measurable stochastic function. Then the sequence f_m is said to converge to g in **probability** if

$$Q\{\omega \in \Omega : |f_m(\omega) - g(\omega)| > \epsilon\} \to 0 \quad \text{as} \quad m \to \infty \quad \forall \epsilon > 0$$

it is said to converge **almost surely** if

$$Q\{\omega \in \Omega : \lim_{m \to \infty} |f_m(\omega) - g(\omega)| \to 0\} \to 0 \quad \text{as} \quad m \to \infty \quad \forall \epsilon > 0$$

Lemma 10.2.2 (Borel-Cantelli) *Suppose $(\Omega, \mathcal{T}, Q)$ is a probability space, and let A_m be a sequence of sets in $\mathcal{T}$. Define*

$$B \;=\; \limsup_{m \to \infty} A_m := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

Suppose

$$\sum_{m=1}^{\infty} Q(A_m) \;<\; \infty \tag{10.5}$$

Then $Q(B) = 0$.

Remark 10.2.3 The set B can also be defined as

$$\begin{aligned} B &= \{\omega : \omega \in \Omega; \sum_{i=1}^{\infty} I_{A_i} = \infty\} \\ &= \{\omega : \omega \in A_n \text{ infinitely often}\} \end{aligned}$$

where I_{A_i} is the indicator function of the set A_i. This lemma, therefore, claims that the set of all ω such that they occur infinitely often in the sequence of sets A_m has measure zero if the sequence $\{Q(A_m)\}$ is summable. Borel-Cantelli lemma leads to the following useful result on almost sure convergence

Lemma 10.2.3 *Suppose $(\Omega, \mathcal{T}, Q)$ is a probability space, that $\{f_m\}_{m \geq 1}$ be a stochastic process on $(\Omega, \mathcal{T}, Q)$, and that $g : \Omega \rightarrow \mathbb{R}$ is measurable. Finally, suppose that*

$$\sum_{m=1}^{\infty} Q\{\omega \in \Omega : |f_m(\omega) - g(\omega)| > \epsilon\} < \infty \; \forall \; \epsilon > 0 \tag{10.6}$$

Then the sequence $\{f_m\}$ converges to g almost surely.

Definition 10.2.8 (Vapnik-Chervonenkis Dimension) *Let $(X, \mathcal{S})$ be a given measurable space, and let $\mathcal{A} \subseteq \mathcal{S}$. A set $S = \{x_1, \cdots, x_n\} \subseteq X$ is said to be **shattered** by $\mathcal{A}$ if, for every subset $B \subseteq S$, there exists a set $A \in \mathcal{A}$ such that $S \cap A = B$. The **Vapnik-Chervonenkis dimension** of $\mathcal{A}$, denoted by VC-dim($\mathcal{A}$), equals the largest integer n such that there exists a set of cardinality n that is shattered by $\mathcal{A}$.*

Remark 10.2.4 In other words, a set S is said to be shattered by a collection of sets $\mathcal{A}$ if for every subset of S, there exists atleast one set $A \in \mathcal{A}$ that uniquely identifies a corresponding subset of S. VC-dim($\mathcal{A}$) in some sense gives us an idea of the largest set whose subsets can be distinguished by the collection of sets $\mathcal{A}$. Now it is easy to see that every set S with cardinality n has 2^n distinct subsets. Therefore, if a collection of sets $\mathcal{A}$ shatters the set S, it must have at least 2^n elements in it to distinguish all the subsets of S.

Example 10.2.1 (Vidyasagar 1997) Let us consider $X = \mathbb{R}$ and a collection of subsets of X, denoted by $\mathcal{A}$ and defined as

$$\mathcal{A} = \{(-\infty, t] : t \in \mathbb{R}\} \tag{10.7}$$

We now show that VC-dim($\mathcal{A}$) = 1. In order to prove it, let us consider $S = \{x_1\}$. This set has two subsets - null set and the set itself. Now, it is easy to see that $(-\infty, t] \cap S = \phi$

for all $t < x_1$ and $(-\infty, t] \cap S = \{x_1\}$ for all $t \geq x_1$. In other words, we have found two distinct sets of sets that represent each subset of S *i.e.*, S is shattered by $\mathcal{A}$. Hence, the VC-dim$(\mathcal{A}) \geq 1$. Let us consider a set with two elements in it, $S = \{x_1, x_2\}$ and assume without loss of generality that $x_1 < x_2$. This set has four subsets - $\phi, \{x_1\}, \{x_2\}, \{x_1, x_2\}$. For every set $A \in \mathcal{A}$ such that $S \cap A = \{x_2\}$, it is also true that $S \cap A = \{x_1\}$. Therefore, $\mathcal{A}$ does not shatter S which implies that VC-dim$(\mathcal{A}) = 1$.

Example 10.2.2 (Vidyasagar 1997) Let us consider $X = [0,1]^2$ and a collection of strictly convex subsets of X, denoted by $\mathcal{A}$. It is possible to show that VC-dim$(\mathcal{A})$ infinite. Consider any set $S = \{x_1, x_2, \cdots, x_n\}$ such that no element of S is a convex combination of rest of the elements. Now, given any subset B of S, we can define a convex hull with the elements of B as its vertices. Clearly, $B \in \mathcal{A}$. Hence, every subset of S is also an element of $\mathcal{A}$, implying that the VC-dim$(\mathcal{A})$ is infinite.

Definition 10.2.9 (Pollard-dimension) *Let $(X, \mathcal{S})$ be a measurable space, and let $\mathcal{F}$ be a set of measurable functions with the range $[0,1]$. A set $S = \{x_1, \cdots, x_n\} \subseteq X$ is said to be **P-shattered** by $\mathcal{F}$ if there exists a real vector $c \in [0,1]^n$ such that, for every binary vector $e \in \{0,1\}^n$, there exists a corresponding function $f_e \in \mathcal{F}$ such that*

$$f_e(x_i) \geq c_i \qquad \text{if } e_i = 1, \text{ and } f_e(x_i) < c_i \text{ if } e_1 = 0. \tag{10.8}$$

*Then the **P-dimension** of $\mathcal{F}$, denoted by P-dim($\mathcal{F}$), is defined as the largest integer n such that there exists a set of cardinality n that is P-shattered by $\mathcal{F}$.*

Remark 10.2.5 P-dimension of sets of functions is equivalent to VC-dimension of sets of sets. Every subset of set S can be denoted by a binary code consisting of n digits, with a zero in the place of elements which are not present and one in the place of elements which are included in the set. For instance, a subset $B := \{x_1, x_3, x_5\}$ can be represented by $e := [1010100000\cdots]$. Hence, every subset is uniquely represented by the vector e. Now the condition presented in the definition of P-dimension ensures that all the functions satisfying them represent one and only one subset of S. Thus, distinguishing all the subsets and P-shattering S.

10.3 Problem Formulation & Solution

The problem of system identification is to fit a given time series data to a set of models chosen *a priori* assuming that the data is generated by a true but unknown model. In any practical situation only a finite data set is available to fit the data. In which case

there are a few pertinent questions that need to be answered. Does the estimated model converge to the best possible model in the model set considered? Does the estimated model converge to the true model as the data length goes to infinity? If so, what is the rate at which it converges to the true model? These are important questions that need to be answered so that an user has idea about the number of data points required to estimate a model with a given accuracy.

Already a few methods have been developed to estimate the rate of convergence of the estimated models to the corresponding best possible models in the model set considered.

In this chapter an attempt is made to solve the same problem in a more general framework using the results from statistical learning theory. In a seminal paper Ljung (1999) has answered some of the questions posed above. He has proved that the estimated models do converge to the true or the best possible models in the model set if the estimated cost function of the prediction errors (defined in a later section) converges uniformly in the model parameters to the true cost function. He also showed that the uniform convergence of the cost function occurs under two assumptions (Ljung 1999):

- The model class consists of uniformly exponentially stable systems
- The parameter vector enters the model in a differentiable manner

In Vidyasagar (1997) a few preliminary results showing the convergence rates for the estimated models based on statistical learning theory are presented. They assume that the signals generated by the process are all stationary and derive certain upper bounds for model convergence for ARMA models. In the current chapter, some of those results are modified to account for the non-stationary signals involved in typical identification problems.

Let us consider the following true process

$$\mathcal{S} \; : \; y(t) = Gu(t) + He(t) \tag{10.9}$$

and assume that the data are generated by the above true process where $G(q)$ is the true process model and $H(q)$ is the true noise model. Also assume that $e(t)$ is gaussian white noise and $e(t), y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Let us suppose that there is an underlying probability space $(\Omega_1, \mathcal{F}_1, Q_1)$ such that

$$e(t, \omega) : \mathbb{R}^+ \times \Omega_1 \to \mathbb{R}$$

For the sake of convenience, we ignore the 'ω' variable in $e(t, \omega)$ and write it as simply $e(t)$ in the rest of this chapter. Now it is possible to equip $e(t)$ with a corresponding Borel σ-algebra $(X_1, \mathcal{T}_1, P_1)$ such that $X_1 \subseteq \mathbb{R}^n$ and $e(t) \in X_1$. In the rest of this chapter

we will be dealing with sequences of data and therefore let us define a measurable space whose underlying set is the cartesian product X_1^∞, consisting of all sequences of the form $\{e(t)\}_{t=1}^\infty$[2] each $e(t) \in X_1$. A **cylinder set** (Vidyasagar 1997) $A \subseteq X_1^\infty$ is a set of the form $\Pi_{t=1}^\infty A_t$ where $A_t \in \mathcal{S}$ for all t, and in addition, $A_t = X_1$ for all except a finite number of indices t. Let $\mathcal{T}_1^\infty$ denote the smallest σ-algebra on X_1^∞ that contains all the cylinder sets. It is now possible to define a corresponding probability measure P_1^∞ on the space $(X_1^\infty, \mathcal{T}_1^\infty)$. Given any cylinder set $A = \Pi_{i=1}^\infty \subseteq X_1^\infty$, define

$$P_1^\infty = \Pi_{t=1}^\infty P_1(A_t).$$

Another important assumption made in the rest of this chapter is that the data is collected under open loop conditions and therefore the input is independent of noise sequence $e(t)$. Input is assumed to be known for all t. The signals encountered in system identification are not all stationary. Some of them are only quasi-stationary necessitating an average over time to obtain time-invariant properties.

Finally, let us assume that we are interested in obtaining the best possible model from the following model class

$$\hat{\mathcal{S}} \; : \; y(t+1|t) = \hat{H}(q,\theta)^{-1}\hat{G}(q,\theta)u(t+1) + (1 - \hat{H}(q,\theta)^{-1})y(t+1) \quad (10.10)$$

where $\theta \in \Theta$ and define the prediction error to be

$$\hat{\epsilon}(t,\theta) = \hat{H}(q,\theta)^{-1}\left[y(t) - \hat{G}(q,\theta)u(t)\right]. \quad (10.11)$$

Now the identification problem is to minimize an objective function which depends on the input-output data denoted by $Z^N = \{y(t), u(t)\}_{t=1}^N$ where N is the length of the data. We choose the standard identification objective function defined by

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N} \hat{\epsilon}(t,\theta,Z^N)^2 \quad (10.12)$$

Since some of the signals involved are non-stationary, $E\hat{\epsilon}(t)^2$ is time-varying and therefore the true cost function is defined as

$$\begin{aligned} \bar{V}(\theta, Z^N) &= \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E[\hat{\epsilon}(t,\theta,Z^N)^2, P_1^\infty] \\ &= \bar{E}[\hat{\epsilon}(t,\theta,Z^N)^2, P_1^\infty] \end{aligned} \quad (10.13)$$

The above limit must exist under the assumptions presented in Chapters 3 and 4. In order to use standard results from statistical theory, we need to define another

[2]It is also assumed that $e(t) = 0$ for $t < 0$.

measurable space independent of $(X_1^\infty, \mathcal{T}_1^\infty, P_1^\infty)$ so that the $\bar{E}$ operator is replaced with an expectation operator over that probability space. Let us assume that the input sequence is generated by filtered white noise, $e'(t)$ which is independent of $e(t)$ defined earlier. Then a corresponding probability space $(X_2^\infty, \mathcal{T}_1^\infty, P_1^\infty)$ for $\{e'(t)\}_{t=1}^\infty$ can be defined. Now it is possible to define the space $(X_1^\infty \times X_2^\infty, \mathcal{F}, P)$ where P and $\mathcal{F}$ are defined as follows:

$$P(A) = P_1^\infty(A)P_2^\infty(A) \qquad \text{since } e(t) \text{ and } e'(t) \text{ are independent of each other} \tag{10.14}$$

and $\mathcal{F}$ consists of all the cylinder sets from the set $X_1^\infty \times X_2^\infty$. With these definitions, we can write

$$\begin{aligned} \bar{V}(\theta) &= \bar{E}[\hat{\epsilon}(t,\theta,Z^N)^2, P_1^\infty] \\ &= E[\hat{\epsilon}(t,\theta,Z^N)^2, P] \end{aligned} \tag{10.15}$$

We are now ready to present the main results of this chapter. The first result ensures convergence of the estimated parameters to the true parameters if the empirical identification objective function converges to the true objective function. That is followed by another result which provides an upper bound on the rate of convergence. This upper bound depends on the P-dimension of a certain set of functions. In the final result an upper bound on the P-dimension is presented. The following result on Uniform Convergence of Empirical Means (UCEM) is a modification of the Theorem presented in Vidyasagar (2001). Unlike Vidyasagar (2001), it is not assumed that the objective function is stationary. Instead, the new probability space defined above, $(X_1^\infty \times X_2^\infty, \mathcal{F}, P)$ is used to make the objective function stationary.

Theorem 10.3.1 *For each $N \geq 1$ and each $\theta \in \Theta$, define*

$$\theta_N^* = \underset{\theta \in \Theta}{argmin}\ V_N(\theta, Z^N) \tag{10.16}$$

and let

$$V^* := \inf_{\theta \in \Theta} \bar{V}(\theta) \tag{10.17}$$

Define the quantity

$$q(N,\epsilon) := P\{\sup_{\theta \in \Theta} |V_N(\theta) - \bar{V}(\theta)| > \epsilon\} \tag{10.18}$$

Now let us assume that $q(t,\epsilon) \to 0$ as $t \to \infty$. Then the following is true

$$P\{V_N(\theta_N^*) > V^* + \epsilon\} \to 0 \quad as \ \ t \to \infty \tag{10.19}$$

In particular, given any $\epsilon, \delta > 0$, choosing N_0 large enough that

$$q(N, \epsilon/3) < \delta \ \forall \ N \geq N_0.$$

Then

$$P\{V_N(\theta_N^*) > V^* + \epsilon\} < \delta \ \forall \ N \geq N_0 \tag{10.20}$$

Proof: Since $q(N, \epsilon) \to 0$ as $N \to \infty$, for any given $\delta > 0$, there exists a N_0 such that for all $N \geq N_0$

$$q(N, \epsilon) \leq \delta \qquad \forall \ N \geq N_0$$

Therefore, with at least probability $1 - \delta$,

$$P\{\sup_{\theta \in \Theta} |V_N(\theta) - \bar{V}(\theta)| \leq \epsilon\}$$

which is equivalent to

$$-\epsilon \leq V_N(\theta) - V(\theta) \leq \epsilon \qquad \forall \ \theta$$

Since the above inequalities are true for all θ, they must also hold for $\theta = \theta^*$.

$$\epsilon + \bar{V}(\theta^*) \geq V_N(\theta^*) \geq V_N(\theta_N^*) \tag{10.21}$$

Hence, with probability of at most δ,

$$V_N(\theta_N^*) > \bar{V}(\theta^*) + \epsilon$$

□

Remark 10.3.1 The condition that $q(N, \epsilon) \to 0$ as $N \to \infty$, is popularly known as Uniform Convergence of Empirical Means (UCEM) in the literature on statistical learning theory. The above result shows that if the set of functions of identification parameters, $\bar{V}(\theta)$ has the UCEM property then minimization of $V_N(\theta, Z^N)$ provides the optimal estimate as the data length tends to infinity. An interesting question to ask is - what type of function sets or model structures have the UCEM property. This is answered in the next two sections.

Before we proceed with presenting the other results of this section, we need to make certain assumptions on the type of models we consider.

A1. Assume that the inputs and outputs belong to a compact set $\mathcal{Z}$.

A2. Assume that the model set consists of only BIBO stable models and that the estimated noise model is inversely stable.

A3. Assume that there exists a constant C, such that

$$V_N(\theta, Z^N) \leq C \quad \text{for all admissible } Z^N \tag{10.22}$$

A4. For use in the proofs of results presented below, let us define

$$\hat{\epsilon}(t, \theta, k) := \hat{W}_u^k u(t) + \hat{W}_y^k y(t) \tag{10.23}$$

where $\hat{W}_u^k$ consists of only the first k impulse response coefficients of $\hat{H}^{-1}\tilde{G}$ and $\hat{W}_y^k$ is similarly defined. In the above definition we have suppressed the argument Z^N for convenience. Most of the results from statistical learning theory that we use in this chapter are valid only for independent data. Unfortunately, when dealing with dynamical data, data are highly correlated. In other words, $\hat{\epsilon}(t, \theta, Z^N)$ are all correlated for different values of t. On the other hand, by virtue of its definition $\hat{\epsilon}(t, \theta, k)$ is a function of only k inputs and outputs. Hence, it is possible to divide Z^N into $[\frac{t}{k}]$ independent sets so that we obtain independent $\hat{\epsilon}(t, \theta, k)$. Also define the following

$$\begin{aligned} V_N^k(\theta, Z^N) &:= \frac{1}{N} \sum_{t=1}^{N} \hat{\epsilon}(t, \theta, k)^2 \\ \bar{V}^k(\theta) &:= E[\hat{\epsilon}(t, \theta, k)^2, P] \end{aligned}$$

We can now define

$$\beta_k = \sup_{\theta \in \Theta} \sup_{u(t), y(t) \in \mathcal{Z}} |\hat{\epsilon}(t, \theta, Z^N)^2 - \hat{\epsilon}(t, \theta, k)^2| \tag{10.24}$$

Assume that β_k is finite for every k and approaches zero as $k \to \infty$. If the predictor is stable, it is easy to see that $\beta_k \to 0$ as $k \to \infty$.

A5. Assume that the set of functions defined by $\mathcal{G}_k = \{\hat{\epsilon}(t, \theta, k)^2 : \theta \in \Theta\}$ has a finite P-dimension, denoted by $p(k)$.

Finally, we present a theorem from Vidyasagar (1997) that is used in proving the next result.

Theorem 10.3.2 (Vidyasagar (1997)) *Suppose that the family of functions, $\mathbb{F} \subseteq [-C, C]^X$, has a finite P-dimension, say p and define*

$$q(m, \epsilon) := P^m\{x \in X^m : \sup_{f \in \mathbb{F}} |\frac{1}{m} \sum_{i=}^{m} f(x_i) - E(f, P)| > \epsilon\}. \tag{10.25}$$

Then

$$q(m,\epsilon) \leq 8\left(\frac{16e}{\epsilon}ln\frac{16e}{\epsilon}\right)^{p} exp(-m\epsilon^2/128C^2) \quad \forall\ m. \tag{10.26}$$

We are know ready to present the second result which provides an upper bound on $q(N,\epsilon)$.

Theorem 10.3.3 (modified from Vidyasagar (2001))
Suppose that the assumptions A1. to A5. are satisfied. Given an $\epsilon > 0$, choose $k(\epsilon)$ large enough that $\beta_k \leq (1-\alpha)\epsilon/2$ for all $k \geq k(\epsilon)$. Then for all $t \geq k(\epsilon)$ we have

$$\begin{aligned} q(N,\epsilon) \leq\ & \frac{l+1}{N}8r\left(\frac{32e}{\alpha\epsilon}ln\frac{32e}{\alpha\epsilon}\right)^{p(k_0)} exp(-(l+1)^2\alpha^2\epsilon^2/512C^2) \\ & + \frac{l}{N}8(k-r)\left(\frac{16e}{\epsilon}ln\frac{16e}{\epsilon}\right)^{p(k_0)} exp(-l^2\alpha^2\epsilon^2/512C^2). \end{aligned} \tag{10.27}$$

where α is a positive constant less than unity.

Proof: Through out this proof we assume that $k \geq k(\epsilon,\alpha) := k_0$. Writing $V_N(\theta, Z^N)$ as $V_N^k(\theta, Z^N) + V_N(\theta, Z^N) - V_N^k(\theta, Z^N) - \bar{V}^k(\theta) + \bar{V}^k(\theta)$, it follows that

$$\begin{aligned} |V_N(\theta,Z^N) - \bar{V}(\theta)| \leq\ & |V_N^k(\theta,Z^N) - \bar{V}^k(\theta)| \\ & + |\left(V_N(\theta,Z^N) - V_N^k(\theta,Z^N)\right) - \left(\bar{V}(\theta) - \bar{V}^k(\theta)\right)| \\ & \text{for any } Z^N \text{ and all } \theta \in \Theta \end{aligned} \tag{10.28}$$

Hence,

$$\begin{aligned} \sup_{\theta\in\Theta}|V_N(\theta,Z^N) - \bar{V}(\theta)| \leq\ & \sup_{\theta\in\Theta}|V_N^k(\theta,Z^N) - \bar{V}^k(\theta)| \\ & + \sup_{\theta\in\Theta}|\left(V_N(\theta,Z^N) - V_N^k(\theta,Z^N)\right) - \left(\bar{V}(\theta) - \bar{V}^k(\theta)\right)| \end{aligned} \tag{10.29}$$

Equivalently, the above inequality can be written as

$$\begin{aligned} P\{\sup_{\theta\in\Theta}|V_N(\theta,Z^N) - \bar{V}(\theta)| > \epsilon\} \leq\ & P\{\sup_{\theta\in\Theta}|V_N^k(\theta,Z^N) - \bar{V}^k(\theta)| > \alpha\epsilon/2\} \\ & + P\{\sup_{\theta\in\Theta}|\left(V_N(\theta,Z^N) - V_N^k(\theta,Z^N)\right) \\ & - \left(\bar{V}(\theta) - \bar{V}^k(\theta)\right)| > (1-\alpha)\epsilon/2\} \end{aligned} \tag{10.30}$$

where $\alpha > 0$ is an adjustment parameter that is introduced to get a better upper bound on the rate of convergence than that in Vidyasagar (2001). The second term on the

right hand side of the above inequality is zero since $\beta_k \leq (1-\alpha)\epsilon/2$. Therefore (10.30) can be rewritten as

$$P\{\sup_{\theta\in\Theta}|V_N(\theta,Z^N)-\bar{V}(\theta)|>\epsilon\} \leq P\{\sup_{\theta\in\Theta}|V_N^k(\theta,Z^N)-\bar{V}^k(\theta)|>\alpha\epsilon/2\} \quad (10.31)$$

It is now possible to partition the data set in such a way that $\hat{\epsilon}(t,\theta,k)$, for any given θ, in a particular partition are all independent. Since the data consists of N data points and they there is a one to one correspondence between the data set to the index set $\{1,2,\cdots,N\}$, let us partition the index set into k subsets as follows (as was done in Vidyasagar (2001),Weyer *et al.* (1999)).

$$\begin{aligned} I_j &:= \{j, j+k, \cdots, j+lk\} \quad \text{for, } j=1,\cdots,r \\ I_j &:= \{j, j+k, \cdots, j+(l-1)k\} \quad \text{for, } j=r+1,\cdots,k \end{aligned}$$

where $l=[Nk]$ and $r=N-lk$. Hence, we can write

$$\begin{aligned} V_N^k(\theta,Z^N) &= \frac{1}{N}\sum_{t=1}^{N}\hat{\epsilon}(t,\theta,k)^2 \\ &= \frac{1}{N}\sum_{j=1}^{r}\sum_{i\in I_j}\hat{\epsilon}(t,\theta,k)^2+\frac{1}{N}\sum_{j=r+1}^{k}\sum_{i\in I_j}\hat{\epsilon}(t,\theta,k)^2 \end{aligned} \quad (10.32)$$

Therefore

$$\begin{aligned} |V_N^k(\theta,Z^N)-\bar{V}^k(\theta)| &= \left|\frac{1}{N}\sum_{j=1}^{r}\sum_{i\in I_j}(\hat{\epsilon}(t,\theta,k)^2-\bar{V}^k(\theta))+\frac{1}{N}\sum_{j=r+1}^{k}\sum_{i\in I_j}(\hat{\epsilon}(t,\theta,k)^2-\bar{V}^k(\theta))\right| \\ &\leq \frac{l+1}{N}\left|\frac{1}{l+1}\sum_{j=1}^{r}\sum_{i\in I_j}(\hat{\epsilon}(t,\theta,k)^2-\bar{V}^k(\theta))\right| \\ &\quad +\frac{l}{N}\left|\frac{l}{l}\sum_{j=r+1}^{k}\sum_{i\in I_j}(\hat{\epsilon}(t,\theta,k)^2-\bar{V}^k(\theta))\right| \end{aligned}$$

Now using theorem 10.3.2, we can write,

$$\begin{aligned} P\{\sup_{\theta\in\Theta}|V_N^k(\theta,Z^N)-\bar{V}^k(\theta)|>\alpha\epsilon/2\} \leq{}& \frac{l+1}{N}8r\left(\frac{32e}{\alpha\epsilon}ln\frac{32e}{\alpha\epsilon}\right)^{p(k_0)}exp(-(l+1)^2\alpha^2\epsilon^2/512C^2) \\ &+\frac{l}{N}8(k-r)\left(\frac{16e}{\epsilon}ln\frac{16e}{\epsilon}\right)^{p(k_0)}exp(-l^2\alpha^2\epsilon^2/512C^2) \end{aligned} \quad (10.33)$$

From (10.31) and (10.33), (10.27) follows.

□

Remark 10.3.2 Theorem 10.3.3 shows that the rate of convergence of the estimated objective function to the true objective function can be quantified using theorem 10.3.1. This upper bound is a function of the P-dimension of $\mathcal{G}_k$. Hence, if the P-dimension is large the rate of convergence for small ϵ is likely to be worse than what it would be for smaller P-dimension. In other words, as expected, the rate of convergence depends on the type of model set. Therefore, it is useful to know how the P-dimension varies with the type of models used for identification. In the next result we provide an upper bound on the P-dimension of the model set considered in theorems 10.3.1,10.3.3.

The following result shows that the pollard dimension of a particular class of functions is unique. It is useful to obtain an upper bound on the P-dimension o model set.

Lemma 10.3.1 *Assume that the identified model is parameterized as follows :*

$$\hat{G}(q,\theta) := \frac{a_0 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}{1 + b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}} \qquad \hat{H}(q,\theta) := \frac{1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}}{1 + d_1 q^{-1} + \cdots + d_{n_d} q^{-n_d}} \quad (10.34)$$

where $\theta = [a_0, \cdots, a_{n_a}, b_1, \cdots, b_{n_b}, c_1, \cdots, c_{n_c}, d_1, \cdots, d_{n_d}]$. *Then* $\hat{\epsilon}(t,\theta,k)$ *is a polynomial in elements of* θ *of maximum order* k.

Proof: Recall that

$$\hat{\epsilon}(t,\theta) = \hat{H}^{-1}(G - \hat{G})u(t) + \hat{H}^{-1}He(t). \quad (10.35)$$

Now the it is easy to see that the k^{th} impulse response coefficient of $\hat{H}^{-1}$ is a polynomial in elements of θ of maximum order $k-1$. Similarly, k^{th} impulse response coefficient of $\hat{G}$ is a polynomial in elements of θ of maximum order k. Therefore, $\hat{\epsilon}(t,\theta,k)$ is a polynomial in elements of θ of maximum order k.

□

Theorem 10.3.4 *Under the assumptions A1. to A6., we have that*

$$P - dim(\mathcal{G}_k) \leq 4r + 2r\log_2 e + 4r\log_2 k \quad (10.36)$$

Proof: Recollect the definition of P-dimension. It involves finding a vector c such that for every binary vector, e

$$\hat{\epsilon}(t,\theta,k)^2 \geq c_t \quad \text{if } e_t = 1 \quad \text{or} \quad \hat{\epsilon}(t,\theta,k))^2 < c_t \quad \text{if } e_t = 0 \quad (10.37)$$

Every function class that satisfies the above inequalities, can be associated with a class of sets using an indicator function. Let us define

$$h_k(\theta, u(t), y(t), c) \;=\; I_{[0,\infty)}[\hat{\epsilon}(t,\theta,k)^2 - c_t] \tag{10.38}$$

where $I_{[0,\infty)}(.)$ is the indicator function for the set $[0,\infty)$. Let us denote the class of sets defined above by

$$\mathcal{A}_k \;:=\; \{\theta \in \Theta : h_k(\theta, u(t), y(t), c) = 1\} \tag{10.39}$$

Using Lemma 10.1 in Vidyasagar (1997), we have

$$P - dim(\mathcal{G}_k) \;=\; VC - dim(\mathcal{A}_k) \tag{10.40}$$

The result in Corollary 10.2 of Vidyasagar (1997) states that if the condition $I_{[0,\infty)}(.) = 1$ can be expressed as a set of s inequalities each of degree no larger than d, then

$$VC - dim(\mathcal{A}_k) \;\leq\; 2r\log_2(4eds) \tag{10.41}$$

It can be easily seen that the above condition is satisfied if

$$\hat{\epsilon}(t,\theta,k)^2 - c_t \geq 0 \tag{10.42}$$

From Lemma 10.3.1, this is a polynomial inequality of maximum degree k^2 *i.e.*, $d = k^2$. Now using (10.41), we have

$$PC - dim(\mathcal{G}_k) = VC - dim(\mathcal{A}_k) \;\leq\; 2r\log_2(4ek^2) \tag{10.43}$$

□

Remark 10.3.3 The above theorem provides an upper bound on the P-dim($\mathcal{G}_k$). This upper bound can in turn be substituted in (10.27) to get an upper bound, which is a function of the data length, the number of parameters and k, on $q(N,\epsilon)$.

10.4 Conclusions

In this chapter, we have modified and generalized a few results presented in Vidyasagar (2001) for linear prediction error models. We have proved that the identified model converges to the true optimal model if the identification objective function has the UCEM property. An upper bound on the rate of convergence of the empirical objective function to the true objective is also derived based on the P-dimension of the model

class. An upper bound on the P-dimension is also derived for the general prediction error models.

The results presented in Vidyasagar (2001) and in this chapter, are useful preliminary results to study the finite sample data properties of the identified models. One of the challenging problems involved in using statistical learning theory to study system identification is due to the temporal correlation between prediction errors. Traditional learning theoretic results assume that the data are independent but on the other hand, system identification involves correlated data. Hence, there is a need to develop new results and modify the existing results in statistical learning theory.

11

Conclusions and Future Work

11.1 Concluding Remarks

The main contribution of this thesis is the analysis and development of a new metric for model quality and its use in an iterative algorithm to implement quasi-adaptive model predictive control. Specific contributions are listed below :

- Theorem and proof that the model predictive control relevant identification is useful if and only if certain amount of bias is expected either in the process or noise model.
- Development of a two step MRI method for identifying an FIR model and a noise model.
- Experimental evaluation of the MRI method on a pilot-scale plant.
- Expression for bias distribution of models identified by minimizing multi-step ahead prediction errors.
- We also showed that noise models with all positive impulse response coefficients lead to lower weighting in the low frequency region and high weighting in the high frequency region on the model bias. The low frequency weighting improves with increasing prediction horizon.
- Using MRI, we have developed an iterative identification and control algorithm to improve closed loop performance.
- We have explored interconnections between identification for MPC and closed loop performance.

- Results on experiment design are provided by deriving an expression for the optimal input spectrum for process identification.
- An approximation of the adaptive dual control problem is developed. A number of interpretations of this approximation are also presented.
- A new model free iterative controller tuning method has been developed.
- Issues involved in nonlinear iterative identification and control are studied using an adaptive control algorithm called backstepping.
- An attempt is made to derive finite sample properties of the estimated models using learning theory.
- An algorithm to obtain unbiased estimates of recycle systems has been developed.
- A new approach based on Expectation Maximization algorithm is developed for identification of multi-rate processes.

11.2 Future Directions

Iterative identification and control is a relatively new area. In this thesis we have concentrated mainly on model predictive controllers. There are a number of interesting issues that are still open and need to be resolved. Some of them are enumerated below.

- It is useful to quantify the advantage(s) of using MRI method over traditional identification methods. It is likely to depend on the signal to noise ratio.
- In the literature on Generalized Predictive Controllers, the noise model is popularly known as T-filter or robustness filter. It would be useful to explore the connections between the MRI filter and the T-filter and study the effect of MRI filter on the robustness of the closed loop.
- The iterative algorithm presented in this thesis assumes that the model predictive controllers do not have any constraints. This may not be an acceptable restriction on the controller. It will be useful to develop similar algorithms for model predictive controllers with constraints.
- Also there is a need to study the impact of receding horizon nature of the model predictive controllers. How much improvement in performance do we get from a receding horizon controller? How does it effect the identification process?
- The method proposed in thesis can probably be described as a non-learning approach. The data collected after each iteration is thrown away and the experiment is repeated. A better approach is probably to find a way to use the data from the

previous experiment. By doing so it might be possible to reduce the variance on the estimated models and also minimize the number of experiments.

- Most of the results presented in this thesis are mainly theoretical with some pilot-scale trials. An industrial application would buttress the arguments provided in this thesis.
- Many results presented in this thesis are based on well known asymptotic results for estimated models. However, in practice it is of interest both to theoreticians and practicians to understand the finite sample properties of the estimated models. We have provided certain preliminary results on finite sample properties of identified models using learning theory. There are a number of interesting issues that are yet to be resolved.
- We have provided an algorithm for identification of delay dominant recycle systems. Identification of recycle systems is similar to that of identification of closed loop systems. However, they pose certain challenges in that the input equivalent signal is not available. There is a need to derive bias expressions for the identified models when the process is not delay dominant. Also, one can explore the possibility of designing appropriate experiments to identify recycle systems.
- Certain preliminary results on identification of multi-rate processes using Expectation Maximization algorithm are presented in this thesis. There are a number of important issues such as informativeness of the data, robustness of the algorithm to initial model guesses, finite sample bias in model etc., which need further study.

Part V

Miscellaneous Topics

12

Identification of Multi-Rate Chemical Processes

In this chapter, we provide a novel iterative identification algorithm for multi-rate sampled data systems. The procedure involves, as a first step, identifying a simple initial model from the multi-rate data. Based on this model, the "missing" data points in the slow sampled measurements are estimated following the expectation maximization approach. Using the estimated missing data points and the original data set a new model is obtained and this procedure is repeated until the models converge. An attractive feature of the proposed method lies in its applicability to irregularly sampled data. An application of the proposed method to an industrial data set is also included[1].

12.1 Introduction

Traditional identification methods assume that the data are sampled at uniformly spaced sample instants. There is extensive literature on identification of processes from such

[1]Portions of this chapter will appear in Gopaluni *et al.* (2003*c*) and Gopaluni *et al.* (2003*b*). A version of this chapter has been submitted for publication (Gopaluni *et al.* 2003*d*)

data (Ljung 1999, Söderström and Stoica 1989). However, in many chemical processes either it is not physically possible to measure certain variables at regular intervals or it is expensive. For instance, estimating the composition of the distillate in a distillation column generally takes a few hours but on the other hand control moves are implemented at much smaller sample intervals. Hence, identification of multi-rate processes is of interest in the chemical industry. Such processes with differing sample times for the measured variables are termed multi-rate processes in the rest of this chapter. Identification of models for multi-rate processes at the fastest sample rate is the subject of this chapter. We refer to the fastest sample rate as the base sample rate and the unavailable data points in the slowly sampled measurements as *missing data*.

One approach to handle multi-rate data is to identify a continuous time model (Sinha and Rao 1991). A discrete approximation of the continuous time model is obtained from the original differential equations describing the process. This approximate model is then identified using the multi-rate data. However, in the chemical industry, the final required model is always discrete. Hence, it is not necessary to identify a continuous time model. Instead a model at the fastest sample time in the process is sufficient for the purpose of prediction. Moreover, most of the available methods for continuous time identification use linear or quadratic interpolation on slowly sampled data.

The other commonly used approach, in the industry, to identification of multi-rate processes is to fill all the missing data points by some form of interpolation between the available data points - often poor linear/quadratic interpolation is used. This type of interpolation does not take into account the variation in the input during the period over which the interpolations are made. There have also been attempts at solving the multi-rate identification problem using lifting techniques (Chen and Francis 1991, Li *et al.* 2001). These methods, in general, do not provide optimal causal models. Also, it is not clear if the models obtained using lifting techniques are statistically optimal. Due to the nature of the lifting operator, even if the original process has a small order, the lifted model, depending on the sampling time, can have a relatively high order. Thus, complicating the problem further.

In this chapter we present a method which uses an initial crude model to estimate the unavailable data points in the slowly sampled variables. The estimated unavailable data points are then used with original data set to identify a new model. From the new model, the missing data is again estimated and the process is repeated until the models converge. The advantage of this method lies in the methodical manner in which the missing data points are estimated instead of using the traditional interpolation methods. Instead of interpolating, the missing data are estimated based on the current estimate of the

process model at each iteration.

This chapter is organized as follows: in section 12.2 assumptions and notation are explained. In section the 12.4, EM algorithm is presented and in section 12.5 a method for identification of linear dynamical systems is developed. In section 12.6 the EM algorithm based identification method is extended to the missing data case. An industrial example is presented in section 12.7 and finally, concluding remarks are made in section 12.8.

12.2 Assumptions and Notation

Let us assume that the true process is of the form

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{12.1}$$

where A, B, C are the system matrices and $x_t \in \mathbb{R}^n$ is the n-dimensional state vector. Assume that $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. w_t and v_t are uncorrelated white noise sequences i.e.,

$$\begin{aligned} E[w_t w_t^T] = Q \;\; ; \;\; E[w_t] = 0 \quad \forall\, t \\ E[v_t v_t^T] = R \;\; ; \;\; E[v_t] = 0 \quad \forall\, t \\ E[w_t v_t] = 0 \qquad \forall\, t \end{aligned} \tag{12.2}$$

Let us represent the time series data from $t = 1$ to $t = N$ of any variable by $(.)_{1:N}$. Through out this chapter, we will use the following notation for the expected values of various variables,

$$x_t^s := E(x_t | Y_{1:s}) \tag{12.3}$$

and

$$P_t^s := E(x_t - x_t^s)(x_t - x_t^s)^T \qquad P_{t,t-1}^s := E(x_t - x_t^s)(x_{t-1} - x_{t-1}^s)^T \tag{12.4}$$

In addition, the following assumptions are made:

Assumptions

A1. Inputs are sampled uniformly every T units of time.

A2. Outputs are irregularly sampled and the maximum sampling times are $T_1, \cdots, T_n$ respectively.

A3. The input sampling time, T, is assumed to be the smallest sampling time *i.e.*,

$$T \leq T_i \qquad \forall\ i \tag{12.5}$$

A4. Assume that the initial state is zero *i.e.*, $x_0 = 0$.

12.3 What is wrong with arbitrary interpolation?

The most common method of dealing with multi-rate data is to interpolate between the known or observed data. This interpolation can be done in a number of ways - zero order hold interpolation, linear interpolation, quadratic interpolation etc. Similar, but implicit interpolation is used in continuous time identification methods too. If there are not too many missing data points or if the missing data points between any two observed data points are small in number then interpolations of above nature may not significantly change the final identified model. However, if the size of missing data points is significantly large compared to the number of observed data points, interpolation of any type can have an adverse effect on the identified model. We show this through simulation and industrial examples in a later section. In fact, there is no guarantee that the identified model is consistent with the real process. Moreover, interpolation based identification methods are not statistically optimal.

In fig.12.1, a typical plot of input-output data is shown. The two diamonds on the output signal indicate sampled output data. It is easy to notice that during that period of time, the input varies in a particular fashion. Now if we ignore this change in input and the input changes immediately before this sample period and perform arbitrary interpolation to fill the missing data points then the filled missing data do not represent the real process. Hence, identification using arbitrary interpolation is akin to "filling the missing data with wrong values and hoping to identify the correct model".

A number of methods are in use in statistical literature to deal with missing data. Among them the most often used methods are (Little and Rubin 1987):

1. **Listwise Deletion**: A large amount of data is deleted in order to create single rate data subset. By deleting data we would lose valuable information that can easily be used with other methods. Also, this method often leads to biased models.
2. **Mean Imputaion**: This method uses the mean value of the parameter to impute the missing value. It is known to lead to biased results of variances and co-variances and hence not a very good method for application.
3. **Maximum Likelihood Estimate (MLE)**: The likelihood function of the data is maximized in order to estimate the model parameters. Those values of the

parameters that maximize the likelihood function are the maximum likelihood estimates. This is a very general method and hence maximum likelihood estimates can be developed for a variety of estimation problems. Asymptotically, these estimates become minimum variance unbiased estimates. However, there are a few disadvantages associated with this method. The computational effort involved may be non-trivial. MLEs can be heavily biased for small data sets and the estimates are usually sensitive to the initial guess.

4. **Expectation Maximization**: Expectation maximization method solves the maximum likelihood estimation problem in an efficient and simpler manner when the data are missing. It is an iterative approach that involves an expectation step and a maximization step. Even though this method simplifies the MLE problem, it has similar properties. The rate of convergence of this method depends on the size of missing data. The larger the missing data set, the slower is the convergence.
5. **Multiple Imputation**: Imputation, the idea of filling in missing data, is an attractive approach to dealing with missing data. Multiple imputation is a Monte Carlo technique in which the missing data are replaced with simulated versions. The EM algorithm and multiple imputation involve averaging a likelihood function. Multiple imputation performs this averaging through Monte-Carlo simulations rather than numerical methods. Multiple imputation provides solutions similar to that of EM algorithm for large sample sizes.

Clearly, the optimal method of filling the missing data points is to use the "true" process model in estimating the missing data. However, we are interested in identifying this true model. An intuitive alternative is to estimate a crude model and use it to estimate the missing data. Once the missing data is estimated, we can create a complete data set that includes both missing and observed data. It is then possible to use prediction error methods on the complete data set. The new model obtained can then be used to re-estimate the missing data. This constitutes the philosophy behind the method proposed in this chapter.

12.4 The EM algorithm

The central idea behind the algorithm presented in this chapter is to pose the multi-rate identification problem in the maximum likelihood framework and solve for the system matrices. The iterative algorithm presented in this section is based on the popular *Expectation Maximization algorithm*(EM algorithm) developed by Dempster *et al.* (1977). Before utilizing this algorithm in identifying multi-rate processes, a brief

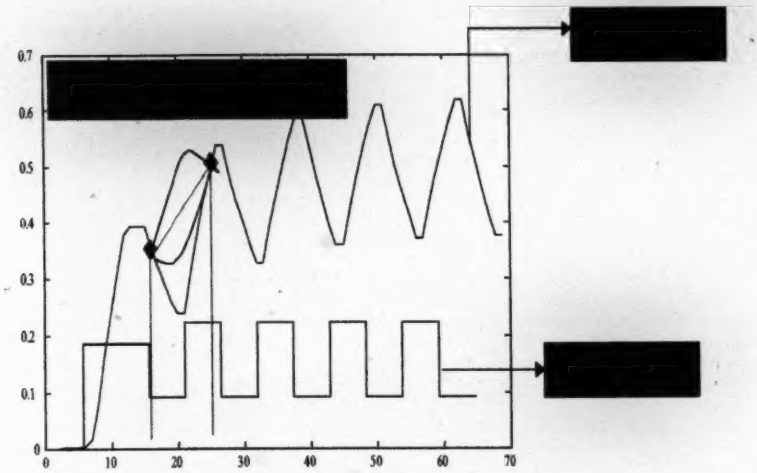


Figure 12.1: Various types of interpolation

summary of the algorithm is presented below.

The EM algorithm addresses the problem of estimating model parameters under the maximum likelihood framework. More often than not, the maximum likelihood function is a complicated nonlinear function of the unknown parameters. Hence, one of the earliest methods proposed for solving for the optimal parameters was to use Newton-Raphson method (Gupta and Mehra 1974). A simpler method based on the EM algorithm was proposed in (Shumway and Stoffer 1982).

Consider the observed data set $y_{1:N}$ and the unobserved data set $x_{1:N}$. Assume that the true data set consists of both $y_{1:N}$ and $x_{1:N}$. Then the idea is to maximize the joint probability density function of the observed data, denoted by $f_y(y_{1:N}|\Theta)$ where Θ is the unknown parameter vector. Let us rewrite the likelihood function including the unobserved data

$$\begin{aligned} f_y(y_{1:N}|\Theta) &= f_y(y_{1:N}|\Theta)\frac{f_x(x_{1:N}|y_{1:N},\Theta)}{f_x(x_{1:N}|y_{1:N},\Theta)} \\ &= \frac{f_{yx}(y_{1:N},x_{1:N}|\Theta)}{f_x(x_{1:N}|y_{1:N},\Theta)} \end{aligned} \tag{12.6}$$

Now applying logarithms on both sides, we get

$$\log f_y(y_{1:N}|\Theta) = \log f_{yx}(y_{1:N},x_{1:N}|\Theta) - \log f_x(x_{1:N}|y_{1:N},\Theta) \tag{12.7}$$

Since the unobserved data, $x_{1:N}$, is random with probability density function, $f_x(x_{1:N}|y_{1:N},\Theta)$, it is not possible to evaluate the right hand side of (12.7). However, we

can find the expected value of the above equation with respect to the unobserved data given the observed data and an estimate of the parameters, Θ^k (usually obtained from a previous iteration) *i.e.*,

$$\begin{aligned} \log f_y(y_{1:N}|\Theta) &= E\left[\log f_{yx}(y_{1:N}, x_{1:N}|\Theta) \mid y_{1:N}, \Theta^k\right] \\ &\quad -E\left[\log f_x(x_{1:N}|y_{1:N}, \Theta) \mid y_{1:N}, \Theta^k\right] \end{aligned} \tag{12.8}$$

The first on the right hand is often called the Q-function *i.e.*,

$$Q(y_{1:N}, \Theta) = E\left[\log f_{yx}(y_{1:N}, x_{1:N}|\Theta) \mid y_{1:N}, \Theta^k\right] \tag{12.9}$$

An important result proved in the papers on convergence of the EM algorithm is

$$E\left[\log f_x(x_{1:N}|y_{1:N}, \Theta^k) \mid y_{1:N}, \Theta^k\right] \geq E\left[\log f_x(x_{1:N}|y_{1:N}, \Theta) \mid y_{1:N}, \Theta^k\right] \ \forall \Theta$$

Now, let us suppose that a new estimate of Θ increases the value of the Q-function, then it is easy to see that the left hand side of (12.8) also increases. Thus increasing the likelihood function of the observed data. Using the above ideas, the EM algorithm can be summarized into a few steps :

- Obtain an initial estimate of the parameter vector, Θ^0.
- Carry out the following steps at each iteration, k, until convergence:
  - **Expectation (E-step)**: Find the expected value of the complete data log likelihood function(Q-function) given the observed data set, $y_{1:N}$ and the previously estimated parameter vector, Θ^k.
  - **Maximization (M-step)**: Maximize the Q-function with respect to the parameter vector

The above steps ensure that the log likelihood function of the observed data increases at every iteration. Therefore, the EM algorithm is guaranteed to converge to a local minimum of the likelihood function. This is an important feature of the EM algorithm. However, there are a few drawbacks associated with any iterative algorithm. EM algorithm can be sensitive to the initial guess and also the rate of convergence can sometimes be extremely slow. In order to avoid problems with bad initial parameter guess, we identify an initial unbiased FIR model of the process. An example illustrating the use of EM algorithm in estimating models from multi rate data is presented below.

Example 12.4.1 Consider an ARX model

$$y(k) = 0.8y(k-1) + 0.3u(k-1) + e(k) \tag{12.10}$$

where $e(k)$ is normally distributed white noise with variance $\sigma_e^2 = 0.01$. Let us assume that output is sampled at every alternate sample instant and that $y(1)$ is known. Then consider the following identification objective function based on squared prediction errors can be used

$$\begin{aligned} V_N(\theta) &:= \frac{1}{N}\sum_{k=1}^{N}\varepsilon(t,\theta)^2 \\ &= \frac{1}{N}\sum_{k=1}^{N}[y(k) - \theta_2 y(k-1) - \theta_1 u(k-1)]^2 \end{aligned} \tag{12.11}$$

where N is the data length and $\theta = [\theta_1 \quad \theta_2]^T$. Since only alternate data points are available, the above objective function can not be evaluated. Instead, it is possible to estimate the expected value of the above objective function given the estimate of θ from the previous iteration, $\hat{\theta}^{(j-1)}$ *i.e.*,

$$E\left[V_N(\theta)|\hat{\theta}^{(j-1)}, Z_N\right] = E[\frac{1}{N}\sum_{k=1}^{N}[y(k) - \theta_2 y(k-1) - \theta_1 u(k-1)]^2] \tag{12.12}$$

where Z_N denotes all the available data. Now let us consider two cases:
Case I: $y(k)$ is known, then

$$\begin{aligned} E\left[y(k) - \theta_2 y(k-1) - \theta_1 u(k-1)\right]^2 &= (y(k) - \theta_1 u(k-1))^2 + \theta_2^2(\hat{\theta}_1^{(j-1)}u(k-2) \\ &\quad +\hat{\theta}_2^{(j-1)}y(k-2))^2 + \theta_2^2\sigma_e^2 \\ &\quad -2(y(k) - \theta_1 u(k-1))\theta_2(\hat{\theta}_1^{(j-1)}u(k-2) \\ &\quad +\hat{\theta}_2^{(j-1)}y(k-2)) \end{aligned} \tag{12.13}$$

Case I: $y(k)$ is unknown, then

$$\begin{aligned} E\left[y(k) - \theta_2 y(k-1) - \theta_1 u(k-1)\right]^2 &= (\hat{\theta}_1^{(j-1)}u(k-1) + \hat{\theta}_2^{(j-1)}y(k-1))^2 \\ &\quad +\sigma_e^2 + (\theta_1 u(k-1) + \theta_2 y(k-1))^2 \\ &\quad -2(\theta_1 u(k-1) + \theta_2 y(k-1))(\hat{\theta}_1^{(j-1)}u(k-1) \\ &\quad +\hat{\theta}_2^{(j-1)}y(k-1)) \end{aligned} \tag{12.14}$$

Using (12.13) and (12.14) in (12.12) it is possible to find the model parameters at the current iteration, j,

$$\theta^{(j)} = \min_{\theta} E\left[V_N(\theta)|\hat{\theta}^{(j-1)}, Z_N\right] \tag{12.15}$$

The iterations are performed until the parameters converge. A plot showing the two parameters in this example and the number of iterations is shown in fig.12.2. The

corresponding decrease in the average prediction error as a function of the iterations is shown in fig.12.3. The estimated model parameters converge to the true parameters despite missing data. In general, the estimates using EM algorithm need not converge to the true parameters with finite data sets. However, the estimated parameters converge to the true parameters asymptotically as the data length increases. On the other hand, the least squares model obtained by interpolating the data is $\hat{\theta_1} = 0.83$ and $\hat{\theta_2} = 0.24$, which is clearly not the true model. There is a small amount of bias in the estimated model using the interpolated data. In general, the estimated models are biassed if arbitrary interpolation methods are used to fill the missing data points.

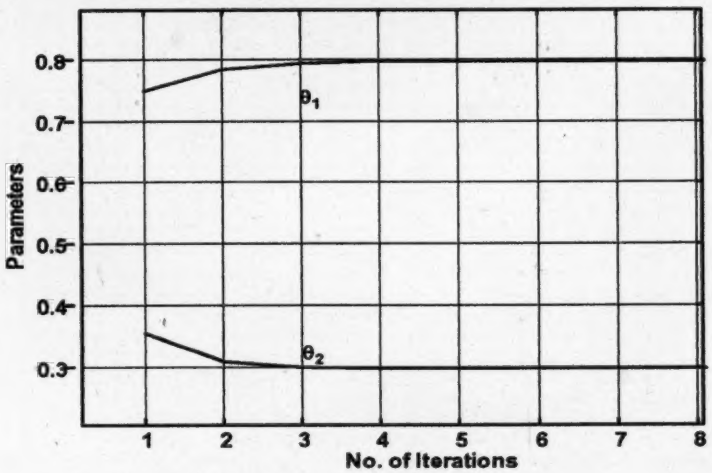


Figure 12.2: Plot of θ_1 and θ_2 as a function of number of iterations

Now it is possible to use this algorithm to estimate the state matrices of a linear dynamical system described in (12.1). As a first step we provide an algorithm to estimate the model from the complete data set *i.e.*, there are no missing data. Even though the data set is complete the states are unknown/unobserved and hence EM algorithm can be utilized. Once a method for identification of single rate systems is developed, it can be extended to the multi-rate systems.

12.5 Estimation of linear dynamical systems

A maximum likelihood framework is adopted in this section to identify the system matrices of (12.1). Two independent sequences of noise enter the dynamical system

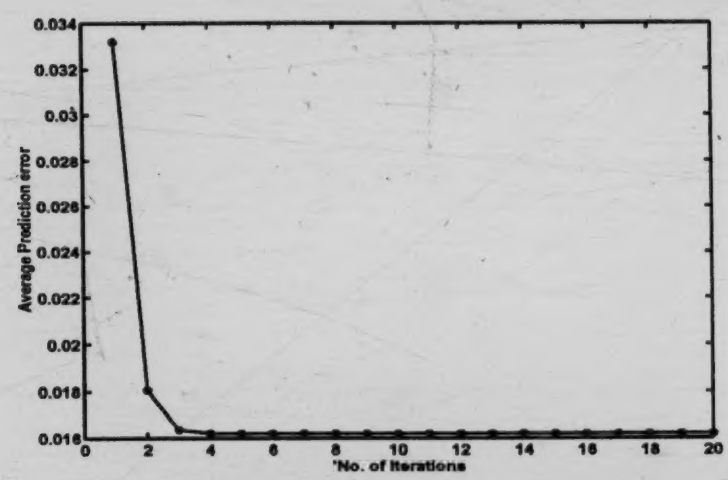


Figure 12.3: Plot of the average prediction error as a function of number of iterations

in (12.1). Hence, the joint log likelihood function of the complete data set can be expressed as

$$\begin{aligned}\log \mathcal{L}(y_{1:N}, x_{1:N}, \Theta) &= \log \mathcal{L}(w_{1:N}, v_{1:N}, \Theta) \\ &= -\frac{N}{2}\log|Q| - \frac{1}{2}\sum_{t=1}^{N}(x_t - Ax_{t-1} - Bu_{t-1})^T Q^{-1}(x_t - Ax_{t-1} - Bu_{t-1}) \\ &\quad -\frac{N}{2}\log|R| - \frac{1}{2}\sum_{t=1}^{N}(y_t - Cx_t)^T R^{-1}(y_t - Cx_t) \qquad (12.16)\end{aligned}$$

where the parameter vector $\Theta = \{A, B, C, Q, R\}$. The Q-function can then evaluated by taking the expectation of the log likelihood function given the observed data and the parameters from the previous iterate (say Θ^k). Let us define the conditional expectation operator E_k as follows

$$E_k(.) = E(.|y_{1:N}, u_{1:N}, \Theta^k) \qquad (12.17)$$

Now using the above expectation operator the Q-function can be evaluated,

$$\begin{aligned}Q(y_{1:N}, \Theta^k, \Theta) &= -\frac{N}{2}\log|Q| - \frac{1}{2}\sum_{t=1}^{N} tr\{Q^{-1}E_k(x_t - Ax_{t-1} - Bu_{t-1})(x_t - Ax_{t-1} - Bu_{t-1})^T\} \\ &\quad -\frac{N}{2}\log|R| - \frac{1}{2}\sum_{t=1}^{N} tr\{R^{-1}E_k(y_t - Cx_t)(y_t - Cx_t)^T\} \qquad (12.18)\end{aligned}$$

where $tr(.)$ denotes the trace of a matrix. At each iteration in the EM algorithm a new estimate of the model is obtained by maximizing $Q(y_{1:N}, \Theta^k, \Theta)$ with respect to Θ *i.e.*,

$$\Theta^{k+1} = \max_{\Theta} Q(y_{1:N}, \Theta^k, \Theta) \tag{12.19}$$

Complete details on obtaining the new estimate, Θ^{k+1} are given in Appendix 12.A .

Example 12.5.1 Consider the following state space model

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 0.3688 & 0.4767 & 0.0114 \\ -0.5976 & 0.6095 & -0.5408 \\ -0.0156 & -0.0686 & 0.0422 \end{bmatrix} x_t + \begin{bmatrix} 0.34 \\ 0.56 \\ 0.78 \end{bmatrix} u_t + w_t \\ y_t &= [1.2 \quad 0.96 \quad 1.5]\, x_t + v_t \end{aligned} \tag{12.20}$$

with the true covariance matrices

$$Q = \begin{bmatrix} 0.0407 & 0.0001 & 0.0015 \\ 0.0001 & 0.0407 & -0.0020 \\ 0.0015 & -0.0020 & 0.0428 \end{bmatrix} \qquad R = 0.398 \tag{12.21}$$

Using the method proposed in this section one can estimate the model parameters. A plot showing the step responses of the true model, a model obtained using the subspace identification method - N4SID and the model obtained using the EM algorithm are shown in fig.12.4. The EM algorithm performs as well as the subspace method. The EM algorithm presented in this section, theoretically, will provide asymptotic unbiased estimates. However, in practice the algorithm may not converge fast enough or if a bad initial guess is given, it may converge to a local maximum. Hence, a good initial guess for the EM algorithm is needed. An unbiased least squares model can be used as the initial guess.

12.6 Estimation with missing data

The strength of EM algorithm lies in being able to estimate asymptotically unbiased models even if a portion of the data is missing. As shown in the previous section, it is possible to use the EM algorithm for identification of models from single rate data sets. However, the computational effort involved in using the EM algorithm is too heavy to warrant this method for single rate identification problems. Moreover, traditional identification methods can provide asymptotically unbiased estimates for single rate data sets. On the other hand, in general, identification methods involving arbitrary interpolations to substitute for missing data result in biased estimates. Thus,

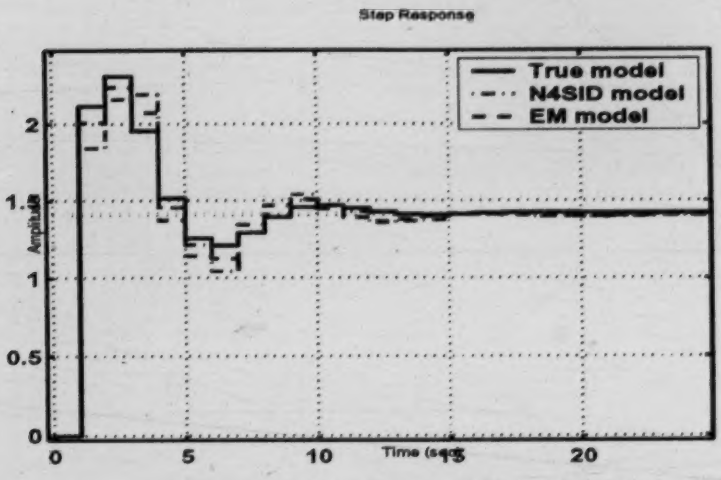


Figure 12.4: Step responses of - the true model, the N4SID model and EM model

necessitating the development of new methods for identification of models from multi-rate data.

It is interesting to note that the EM algorithm presented in the previous section for identification of linear dynamical systems from single rate data, treats the states as unknown/missing data. Hence, it is possible to extend the same algorithm to include the case of missing data in the outputs by making appropriate changes to the Kalman filter and the Kalman Smoother presented in the Appendix. Full details regarding these modifications can easily be derived along the lines of the arguments given in (Shumway and Stoffer 2000).

The procedure can be summarized as follows:

Step 1: Obtain an initial estimate of the model. For instance, it is easy to obtain an FIR model.

Step 2: Estimate the missing data points using the initial estimate of the model. This can be done using the Kalman Filter and the Kalman Smoother.

Step 3: Predict all the missing data points using the current model.

Step 4: Using the true and the estimated missing data points identify a new model by minimizing the Q-function.

Step 5: Repeat the above steps until convergence.

Example 12.6.1 The process in example 12.5.1 is used to generate multi-rate data. The input is sampled every second and the output is sampled every four seconds.

Initially, a model is identified using linearly interpolated data and the N4SID algorithm. Then the proposed method is used on the same data set without interpolating the data. The step responses of both the models are shown in fig.12.5. Clearly, the EM based method outperforms the N4SID method.

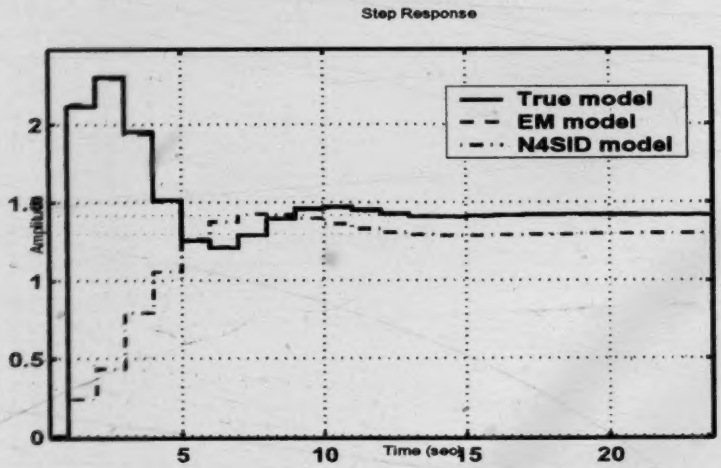


Figure 12.5: Step responses of - the true model, the N4SID model and EM model; Output sampled every four seconds

12.7 Industrial Application

In this application, modelling of a mechanical pulp bleaching process at Millar Western, Whitecourt, Canada is shown. The system has four manipulated inputs, two measured disturbance variables and one output. The output, pulp brightness, is an irregularly measured quality variable (distribution of sampling intervals are provided in fig.12.6). The manipulated inputs are chemical add-rates (Peroxide and Caustic) to two towers. The measured disturbances are two wood quality variables (Aspen and Freeness). All inputs are sampled every 10 minutes). The process is known to be a time-delay dominant recycle process. The step responses of the true model have large delay, fast dynamics and recycle characteristics. In general, the presence of a recycle stream can significantly alter the dynamics of a process ((Morud and Skogestad 1994),(Kwok *et al.* 2001)). This is especially evident when the process dynamics are faster than the time-delay effects in the process. For example, a step change in one of the inputs in a time-delay dominant recycle process, can cause a staircase-like structure in the output as shown in the fig.12.7.

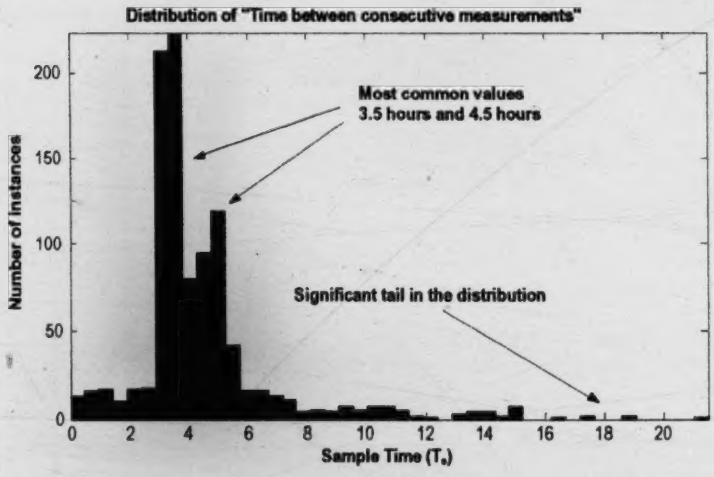


Figure 12.6: Distribution of "time between consecutive measurements"

When the time-delay in the system is greater than the settling time, including lagged inputs (the extra lags being equal to the sum of the delays in the forward path and the recycle path) as predictors can give a better model. In this particular modelling exercise, there was no provision for performing dynamic tests to aid model development. Hence we used routine operating data to perform model identification. The routine operating data has enough excitation in the form of grade changes to justify the exercise of model identification using this data.

We used the proposed method based on the EM algorithm for identifying the model. The predictions based on the EM model (without interpolation) and N4SID model (with zero order hold interpolation) are presented in fig.12.8. The models shown have been adjusted taking the recycle characteristics into account. Hence, only the forward path dynamics are shown. Though it appears that the EM model and the N4SID model perform comparably well for the given data set, it is clear from the step responses (fig.12.9) that the EM model is representative of the true process dynamics (fast dynamics).

12.8 Conclusions

An identification approach for multi-rate data, based on the Expectation Maximization approach is presented. Unlike, traditional identification methods for multi-rate data,

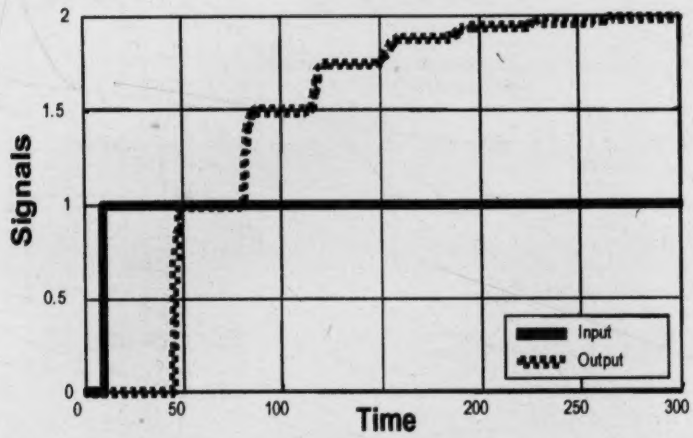


Figure 12.7: Step response of a Delay-dominant recycle process

the proposed method does not use interpolation. An attractive feature of the algorithm is that it can easily handle irregularly sampled data. It leads to asymptotically unbiased estimates of the true model. However, the proposed method is sensitive to the initial guess and is computationally intensive.

12.A Appendix : system matrices at each iteration - Full data case

Let us first evaluate the second term in (12.18)

$$\begin{aligned} T_2 &:= -\frac{1}{2}\sum_{t=1}^{N} tr\{Q^{-1}E_k(x_t - Ax_{t-1} - Bu_{t-1})(x_t - Ax_{t-1} - Bu_{t-1})^T\} \\ &= -\frac{1}{2}tr\{Q^{-1}\left[\Phi_1 + A\Phi_2A^T + B\Phi_4B^T + 2\Phi_3A^T + 2\Phi_5B^T - 2A\Phi_6B^T\right]\} \end{aligned}$$

where

$$\begin{aligned} \Phi_1 &:= \sum_{t=1}^{N} E[x_t x_t^T] \qquad \Phi_2 := \sum_{t=1}^{N} E[x_{t-1}x_{t-1}^T] \\ \Phi_3 &:= \sum_{t=1}^{N} E[x_t x_{t-1}^T] \qquad \Phi_4 := \sum_{t=1}^{N} u_{t-1}u_{t-1}^T \\ \Phi_5 &:= \sum_{t=1}^{N} x_t^N x_{t-1}^T \qquad \Phi_6 := \sum_{t=1}^{N} x_{t-1}^N u_{t-1}^T \end{aligned} \tag{12.A -1}$$

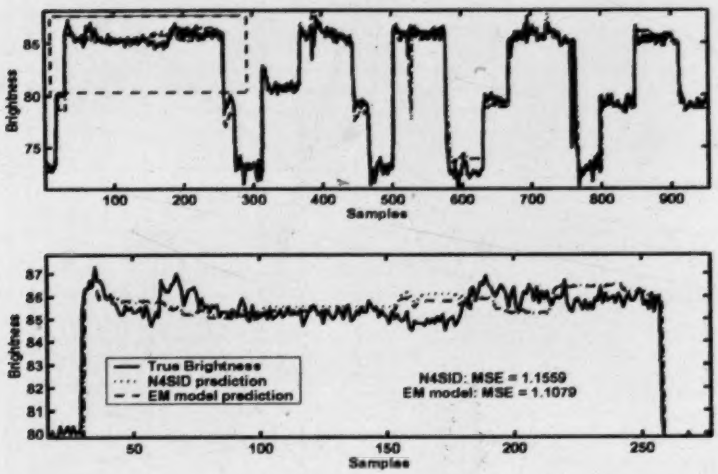


Figure 12.8: Comparison of N4SID and EM model predictions with actual brightness

All the expectations are evaluated using the previous model estimate *i.e.*, using $\Theta^k = \{A^k, B^k, C^k, Q^k, R^k\}$. Observe that this is the only term in (12.18) that depends on the system matrices A and B. Now it is straightforward to differentiate the above expression to obtain the optimal estimates of the system matrices at $(k+1)$th iteration.

$$\begin{aligned} A^{k+1} &= [\Phi_3 - \Phi_5\Phi^{-1}\Phi_6^T][\Phi_2 - \Phi_6\Phi_4^{-1}\Phi_6^T]^{-1} \\ B^{k+1} &= [\Phi_5 - A\Phi_6]\Phi_4^{-1} \end{aligned} \qquad \text{(12.A -2)}$$

Similarly, we can differentiate the first two terms to obtain the optimal new estimate of the covariance matrix, Q^{k+1}

$$Q^{k+1} = \frac{1}{N}\left[\Phi_1 + A^k\Phi_2A^{kT} + B^k\Phi_4B^{kT} + 2\Phi_3A^{kT} + 2\Phi_5B^{kT} - 2A^k\Phi_6B^{kT}\right]$$

The fourth term in (12.18) can similarly be evaluated,

$$\begin{aligned} T_4 &:= -\frac{1}{2}\sum_{t=1}^{N} tr\{R^{-1}E_k(y_t - Cx_t)(y_t - Cx_t)^T\} \\ &= -\frac{1}{2}tr\{R^{-1}\left[\sum_{t=1}^{N} y_ty_t^T + C\Phi_1C^T - 2C\sum_{t=1}^{N} x_t^Ny_t^T\right]\} \\ &:= -\frac{1}{2}tr\{R^{-1}\left[\Phi_7 + C\Phi_1C^T - 2C\Phi_8\right]\} \end{aligned} \qquad \text{(12.A -3)}$$

Since the matrix C appears only in T_4, the new estimate of C, C^{k+1} can be obtained by differentiating T_4 and equating it to zero.

$$C^{k+1} = \Phi_8^T\Phi_1^{-1} \qquad \text{(12.A -4)}$$

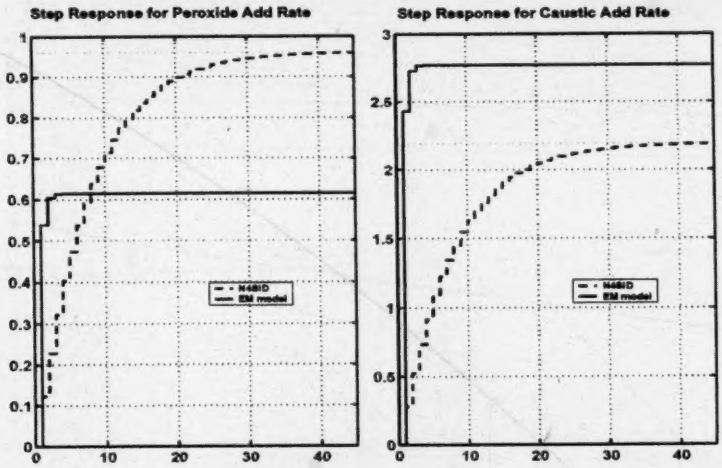


Figure 12.9: Step responses of the N4SID and EM models

In order to obtain an expression for the optimal value of R at the current iteration, we must differentiate the third and fourth term in (12.18). The new estimate of R can be shown to be

$$R^{k+1} = \frac{1}{N}\left[\Phi_7 - \Phi_8^T \Phi_1^{-1} \Phi_8\right] \qquad \text{(12.A -5)}$$

Now we must evaluate all Φ_is. In order to do that, we need to use Kalman filter and a Kalman smoother. Expressions for Kalman filter and Kalman smoother are provided below : (c.f.Shumway and Stoffer (2000)).

Theorem 12.A .1 (Kalman Filter) *For the state space model in* (12.1)*, with initial conditions* x_1^0 *and* P_1^0 *for* $t = 1, \cdots, N$*,*

$$x_{t-1}^t = Ax_t^{t-1} + Bu_t + K_t(y_t - Cx_t^{t-1}) \qquad \text{(12.A -6)}$$

$$P_{t+1}^t = [A - K_tC]P_t^{t-1}[A - K_tC]^T + Q + K_tRK_t^T \qquad \text{(12.A -7)}$$

where the new gain matrix is given by

$$K_t = [AP_t^{t-1}C^T][CP_t^{t-1}C^T]^{-1} \qquad \text{(12.A -8)}$$

Theorem 12.A .2 (Kalman Smoother) *For the state space model in* (12.1)*, with initial conditions* x_N^N *and* P_N^N *obtained using the Kalman Filter, for* $t = N, N-1, \cdots, 1$*,*

$$x_{t-1}^N = x_{t-1}^{t-1} + J_{t-1}(x_t^N - x_t^{t-1}) \qquad \text{(12.A -9)}$$

$$P_{t-1}^N = P_{t-1}^{t-1} + J_{t-1}(P_t^N - P_t^{t-1})J_{t-1}^T \qquad \text{(12.A -10)}$$

where

$$J_{t-1} = P_{t-1}^{t-1} A^T [P_t^{t-1}]^{-1} \qquad (12.A\text{-}11)$$

13

Identification of Delay Dominant Processes with Recycle Streams

In this chapter, we present an identification method for delay dominant recycle systems. Identification of recycle systems is similar to that of closed loop systems. However, identification of recycle systems poses certain challenges in that the input equivalent signal in the closed loop identification is not available for recycle systems. We show that consistent estimates of the forward and recycle models can be obtained for delay dominant recycle systems without having access to the input-equivalent signal[1].

Keywords: Recycle systems, system identification

13.1 Introduction

Chemical processes which operate with material and energy recycle have attracted a lot of attention recently because of their dynamics and the consequent challenges they pose from a control point-of-view (Morud and Skogestad 1996). Research on the empirical model identification (using system identification routines), dynamics and control of recycle processes is of great practical importance because of the wide use of these systems in the process industry. While most of the work published on recycle processes

[1]This chapter has been submitted for publication in Automatica (Gopaluni *et al.* 2003*e*)

concentrates on studying the dynamics and control of recycle processes, there have been relatively few publications which address the challenges in identifying dynamic models for recycle systems (Kwok *et al.* 2001).

Recycle systems consist of a forward path model and a recycle model with either a negative or a positive feedback (Denn and Levi 1982, Morud and Skogestad 1996). Hence, recycle systems are similar to closed loop systems. The dynamic behavior of recycle systems can be totally different from the that of systems with no recycle. Recycle streams generally change the poles and zeros of the forward path leading to different phenomena such as slow response, stair-case like step response or even instability. Luyben (1994) has shown that recycle streams can sometimes lead to a "snowball effect" especially for certain control configurations. The behavior of recycle systems has been well studied in the literature and is also known that due to their atypical behavior special control algorithms are needed to achieve good closed loop performance.

If both the forward and recycle models are available, it is possible to achieve good closed loop performance by designing appropriate controllers (Lakshminarayanan and Takada 2001, Samyudia *et al.* 2000). Given the models, it is possible to remove the recycle effect on the forward model and design an appropriate controller (Scali and Ferrari 1999). Chodavarapu and Zheng (2001) present a method for tuning PID controllers with very little knowledge of the recycle model characteristics. One of the methods suggested by them for identifying the forward model is to remove the recycle stream temporarily for identification. However, in a real industrial setting it is not practical to remove the recycle stream.

Generally, chemical processes exhibit large time delays in the recycle streams. Hence, in this chapter the focus is on time delay dominant recycle systems. When identifying models for recycle processes with dominant time delays it is important to take the complexity of the model structure into account. These systems have staircase-shaped step response. Hence attempts to use black-box identification techniques can lead to the identification of inconsistent models which approximate the staircase responses using high-order step responses. Considering this behavior of typical delay dominant recycle systems, we outline an approach which can provide consistent estimates of both the forward and recycle models.

This chapter is divided into five sections. Section 13.1 gives an overview of the literature on recycle systems and provides certain motivational arguments for studying identification of recycle systems. In section 13.2 the problem of identification of recycle systems is formulated. A method for consistent estimation of forward and recycle models is presented in section 13.3. In section 13.4 some concluding remarks are given in the

final section.

13.2 Problem Formulation

System identification is a well developed area of research and there are a number of books on the techniques involved in identification of consistent models from process data (Ljung 1999, Söderström and Stoica 1989). In the traditional system identification methods, open loop process data is used for estimating unbiased models. On the other hand, there are also methods for identifying models from closed loop data (Forssell and Ljung 1999, Huang and Shah 1997). The problem in closed loop identification is to estimate both the process model and the controller using the setpoint and process input/output data. This can be done in a number of ways *viz.*, two step closed loop identification method, joint input-output identification and projection methods (Forssell and Ljung 1999).

Even though the problem of identification of recycle systems reduces to that of closed loop systems, recycle systems pose certain challenges in identification. Traditional closed loop identification methods utilize three signals namely - set point, process input and process output. On the other hand, in general, for recycle systems process input equivalent signal is not available. Hence, in general, it is not possible to obtain consistent estimates of the forward and recycle models if the input equivalent signal is not available.

A typical recycle system is shown in fig.13.1 where $G_f(q)$ is the true forward process model and $G_r(q)$ is the true recycle model and $H(q)$ is the noise model. We represent the process input by $u(t)$, process output by $y(t)$ and the noise, represented by $e(t)$, is assumed to be normally distributed and white. For the sake of simplicity the process is assumed to be SISO. q represents the forward shift operator. The estimated model consists of an estimated process model represented by $\hat{G}_f(q,\theta)$, an estimated recycle model represented by $\hat{G}_r(q,\theta)$ and an estimated noise model, $\hat{H}(q,\theta)$ where θ is the parameter vector. We assume that $\hat{H}$ and its inverse are both stable and that all the inputs and the outputs are quasi-stationary signals. We also assume that $\hat{G}$ and $\hat{H}$ are independent of the data length by defining the identified models as

$$\hat{G} = \lim_{N\to\infty} \hat{G}^N \tag{13.1}$$

$$\hat{H} = \lim_{N\to\infty} \hat{H}^N \tag{13.2}$$

where $\hat{G}^N$ and $\hat{H}^N$ are models identified from a data of length N.

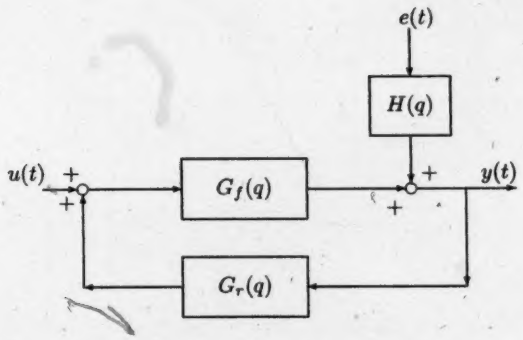


Figure 13.1: Block diagram of a typical recycle system

Let the true model be (see fig.13.1)

$$\begin{aligned} y(t) &= G_f u(t) + G_f G_r y(t) + He(t) \\ &:= Pu(t) + Qy(t-d) + He(t) \\ &:= \Gamma U(k) + He(t) \end{aligned} \tag{13.3}$$

P and Q are defined in an obvious way. $d = d_f + d_r$ is the sum of the forward path delay, d_f and the recycle path delay, d_r and $U(t) = [u(t) \quad y(t-d)]^T$. We assume that $u(t)$ and $e(t)$ are independent. The following additional notation is introduced for convenience.

$$\tilde{\Gamma}(q) := \Gamma - \hat{\Gamma} \quad \text{and} \quad \tilde{H}(q) := H - \hat{H}. \tag{13.4}$$

$\tilde{P}$ and $\tilde{Q}$ are similarly defined. The data collected from identification experiments on the real process are denoted by

$$Z^N = \{u(1), y(1), \cdots, u(N), y(N)\}. \tag{13.5}$$

Example 13.2.1 A typical recycle system and its dynamic behavior is illustrated through this example. Consider the following recycle process

$$G_f = \frac{0.2}{1 - 0.1q^{-1}} q^{-4} \qquad G_r = \frac{0.9}{1 - 0.5q^{-1}} q^{-26}. \tag{13.6}$$

Both forward and recycle models have relatively fast dynamics and the total delay in the recycle loop is much larger than the settling time of either model. The step response of this process is shown in fig.13.2. The stair-case like step response is due to the large delay in the recycle stream and the comparatively fast dynamics in the forward and recycle streams.

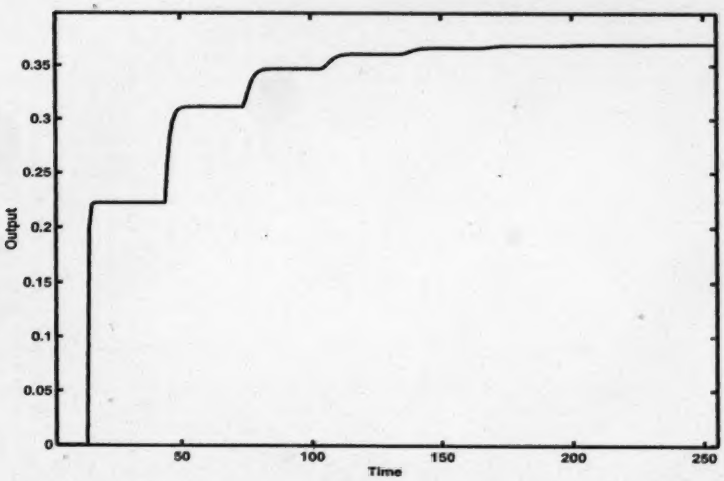


Figure 13.2: A typical stair-case like step response of a recycle system

13.3 Estimation of forward and recycle models

13.3.1 Asymptotic consistency

In this section we show that by treating lagged output as one of the inputs, it is possible to obtain consistent estimates of the forward and recycle models for delay dominant processes. Consider the model equation in (13.3). Then the one-step ahead prediction error objective function for identification can be written as

$$\begin{aligned} V_N(\theta, Z^N) &= \frac{1}{N}\sum_{t=1}^{N}(y(t+1) - \hat{y}(t+1))^2 \\ &:= \frac{1}{N}\sum_{t=1}^{N}\varepsilon(t,\theta)^2 \end{aligned} \tag{13.7}$$

where $\hat{y}(t+1)$ is the optimal one-step ahead predictor given by

$$\hat{y}(t+1) = \hat{H}^{-1}\hat{\Gamma}U(t+1) + (1 - \hat{H}^{-1})y(t+1). \tag{13.8}$$

Then the prediction error estimate is given by

$$\hat{\theta}_N = \underset{\theta \in D_{\mathcal{M}}}{\arg\min}\ V_N(\theta, Z^N) \tag{13.9}$$

Theorem 13.3.1 *Consider the model in* (13.3) *and the estimation method in* (13.9). *Also assume that*

$$\tilde{P} := q^{-d_f}\sum_{k=0}^{n} p_k q^{-k} \qquad \tilde{H} := \sum_{k=0}^{m} h_k q^{-k}$$

where $d \geq n+1 \geq 1$ *and* $d \geq m \geq 1$. *Then under the assumptions presented in the previous section*

$$\hat{\theta}_N \rightarrow D_c = \underset{\theta \in D_{\mathcal{M}}}{arg\ min} \int_{-\pi}^{\pi} |\hat{H}|^{-2}|\tilde{P}|^2\Phi_u + |\hat{H}|^{-2}|\tilde{Q}|^2\Phi_{y_d} + |\hat{H}|^{-2}|\tilde{H}|^2\Phi_e d\omega \quad (13.10)$$

with probability one. Hence, it is possible to obtain consistent estimates of $\tilde{P}$ *and* $\tilde{Q}$.

Proof: As the data length, N, tends to infinity, the prediction error identification objective function in (13.7) converges almost surely to (Forssell and Ljung 1999)

$$\lim_{N\to\infty} V_N(\theta, Z^N) \rightarrow \bar{V}(\theta)$$
$$:= \frac{1}{2\pi}\int_{-\pi}^{\pi} tr\left(\begin{bmatrix} \tilde{\Gamma} & \tilde{H} \end{bmatrix} \begin{bmatrix} \Phi_U(\omega) & \Phi_{Ue}(\omega) \\ \Phi_{eU}(\omega) & \Phi_e(\omega) \end{bmatrix} \begin{bmatrix} \tilde{\Gamma} \\ \tilde{H} \end{bmatrix}^* (\hat{H} * \Phi_e * \hat{H}^*)^{-1} \right) d\omega$$

This objective function can be expanded to

$$\bar{V}(\theta) :=$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} tr\left(\begin{bmatrix} \tilde{P} & \tilde{Q} & \tilde{H} \end{bmatrix} \begin{bmatrix} \Phi_u(\omega) & \Phi_{uy_d}(\omega) & \Phi_{ue}(\omega) \\ \Phi_{y_du}(\omega) & \Phi_{y_d}(\omega) & \Phi_{y_de}(\omega) \\ \Phi_{eu}(\omega) & \Phi_{ey_d}(\omega) & \Phi_e(\omega) \end{bmatrix} \begin{bmatrix} \tilde{P} \\ \tilde{Q} \\ \tilde{H} \end{bmatrix}^* \right.$$
$$\left. (\hat{H} * \Phi_e * \hat{H}^*)^{-1} \right) d\omega \quad (13.11)$$

where y_d is the delayed output, $y(t-d)$. Notice that the minimum of this objective function occurs at the true parameters of the true models *i.e.*, when $\hat{P} = P$, $\hat{Q} = Q$ and $\hat{H} = H$. If the spectral matrix at the center is non-diagonal, it is not possible to guarantee unbiased estimates of the process and noise models unless their correct structures are known. Typically, guessing the true structure of the noise model is very difficult and as a result, often, minimization of the above objective function leads to biased estimates. Let us take a closer look at the off-diagonal terms in the spectral matrix:

Term I: Assume that Φ_e is constant and that $\hat{H} = 1$ *i.e.*, Output-Error structures

are considered. Without loss of generality, assume that the input is white and hence the input spectrum is independent of frequency. Then one of the off-diagonal terms in the spectral matrix is

$$\begin{aligned} T_1 &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{H}^{-1}\tilde{P}\Phi_{uy_d}\tilde{Q}^*(\hat{H}^{-1})^*\Phi_e d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{P}\Phi_{uy_d}\tilde{Q}^*\Phi_e d\omega \end{aligned} \tag{13.12}$$

Now note that

$$\begin{aligned} y(t-d) &= \frac{P}{1-Qq^{-d}}q^{-d}u(t) + \frac{Hq^{-d}}{1-Qq^{-d}}e(t) \\ &:= q^{-d-d_f}\sum_{k=0}^{\infty} g_k q^{-k}u(t) + \frac{Hq^{-d}}{1-Qq^{-d}}e(t) \end{aligned} \tag{13.13}$$

for some impulse response coefficients, g_k. Therefore,

$$\begin{aligned} \Phi_{uy_d}(\omega) &= (\Phi_{y_du}(\omega))^* \\ &= e^{i\omega(d+d_f)}\sum_{k=0}^{\infty} g_k e^{ik\omega}\Phi_u(\omega) \end{aligned} \tag{13.14}$$

Let us consider the case where $\tilde{P} = q^{-d_f}\sum_{k=0}^{n} p_k q^{-k}$ and define $\tilde{Q} = \sum_{k=0}^{\infty} q_k q^{-k}$ for some impulse response coefficients p_k and q_k. Then

$$\begin{aligned} T_1 &:= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{-id_f\omega}\sum_{k=0}^{n} p_k e^{-ik\omega}\right)\left(e^{i(d+d_f)\omega}\sum_{k=0}^{\infty} g_k e^{ik\omega}\right)\left(\sum_{k=0}^{\infty} q_k e^{ik\omega}\right)\Phi_e\Phi_u d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{id\omega}\sum_{k=0}^{n} p_k e^{-ik\omega}\right)\left(\sum_{k=0}^{\infty} g_k e^{ik\omega}\right)\left(\sum_{k=0}^{\infty} q_k e^{ik\omega}\right)\Phi_e\Phi_u d\omega \end{aligned} \tag{13.15}$$

Observe that if $d \geq n+1$, then the integrand in the above expression can be expressed as a summation of infinite exponentials with positive powers *i.e.*,

$$T_1 := \sum_{k=1}^{\infty}\frac{\Phi_e\Phi_u}{2\pi}\int_{-\pi}^{\pi} s_k e^{ik\omega}d\omega \tag{13.16}$$

for some constants s_k. The integral is zero for all positive k. Therefore, T_1 is zero if $d \geq (n+1)$ and

$$\tilde{P} = q^{-d_f}\sum_{k=0}^{n} p_k q^{-k} \tag{13.17}$$

Term II: Now consider the term involving $\Phi_{y_d e}$. This term can be written as

$$\begin{aligned} T_2 &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{H}^{-1}\tilde{Q}\Phi_{y_d e}\tilde{H}^*(\hat{H}^{-1})^*\Phi_e d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{Q}\Phi_{y_d e}\tilde{H}^*\Phi_e d\omega \end{aligned} \tag{13.18}$$

Let us now assume that

$$\tilde{H} := \sum_{k=0}^{m} h_k q^{-k}$$

i.e., the error in the impulse response coefficients of the noise model approaches zero after m impulse response coefficients. Then from (13.13),

$$\begin{aligned} \Phi_{y_d e} &= \frac{H(e^{i\omega})e^{-id\omega}}{1 - Q(e^{i\omega})q^{-d}}\Phi_e(\omega) \\ &:= e^{-id\omega}\sum_{k=0}^{\infty} r_k e^{-ik\omega}\Phi_e(\omega) \end{aligned}$$

Hence,

$$\begin{aligned} T_2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=0}^{\infty} q_k e^{-ik\omega}\right)\left(e^{-id\omega}\sum_{k=0}^{\infty} r_k e^{-ik\omega}\right)\left(\sum_{k=0}^{m} h_k e^{ik\omega}\right)\Phi_e^2(\omega)d\omega \\ &:= \sum_{k=1}^{\infty}\frac{\Phi_e\Phi_e}{2\pi}\int_{-\pi}^{\pi}\nu_k e^{-ik\omega}d\omega \end{aligned} \tag{13.19}$$

The above integral is again zero for all positive k. Due to the assumption that the input is not correlated with the noise, $\Phi_{eu}(\omega) = 0$ at all frequencies. Hence, the result in (13.10) follows.

□

Remark 13.3.1 The conditions on the forward model are clearly satisfied if it is a pure gain model with delay. They are approximately satisfied when the impulse response coefficients of $\tilde{P}$ decay very fast compared to the length of the total delay, d. It is straightforward to show that other cross terms in the spectral matrix are also zero under these conditions. If all the cross terms are zero, then the above objective function reduces to

$$\bar{V}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{H}|^{-2}|\tilde{P}|^2\Phi_u + |\hat{H}|^{-2}|\tilde{Q}|^2\Phi_{y_d} + |\hat{H}|^{-2}|\tilde{H}|^2\Phi_e d\omega \tag{13.20}$$

From this expression, it is clear that even if the noise model is fixed to unity, the minimum occurs at $\hat{P} = P$ and $\hat{Q} = Q$ provided the model set includes the true model. Hence, the above identification algorithm results in unbiased estimates for delay dominant recycle processes.

13.4 Conclusions

We have shown that it is possible to obtain consistent estimates of both forward and recycle models for delay dominant recycle systems. This is achieved by treating a delayed output as one of the inputs.

A

Appendix : Cramer's rule for nonsquare-nonsingular matrices

This is an interesting problem that I solved in my second term at the University. This problem was proposed by Laksh, another Ph.D student of Dr.Shah and was published in the challenge problems section of the American Mathematical Monthly. I wanted to send my solution to them but I found that there was already a solution published in the same journal. In fact, I must admit that the published solution is much more elegant than mine (a "lay man's" solution).

Problem:

Given $AX = B$, where A is an $m \times n$ matrix, X is an $n \times r$ matrix and B is an $n \times r$ matrix, prove that

$$X_{ij} = \frac{\left[\det(A^*_{ij}A^T) - \det(A^0_i A^{0^T}_i)\right]}{\det(AA^T)}$$

where A^*_{ij} is obtained by replacing i^{th} column of A by j^{th} column of B and A^0_i is obtained by deleting the i^{th} column of A.

Proof:

We will prove the above result for a vector b instead of a matrix B. It is easy to see that the result automatically follows if it is true for a vector b.

We need to prove that the i^{th} element of X is :

$$X_i = \frac{\det(A_i^* A^T) - \det(A_i^0 (A_i^0)^T)}{\det(AA^T)} \tag{A.1}$$

Where $AX = b$. A is an $m \times n$ matrix with $m < n$. b is an $m \times 1$ vector and X is an $n \times 1$ vector

$$\begin{aligned} X &= A^T(AA^T)^{-1}b \quad \text{(Common least squares method)} \\ &= \frac{A^T \text{adj}(AA^T)b}{\det(AA^T)} \end{aligned} \tag{A.2}$$

Therefore the ith element of X is

$$X_i = \frac{a_i^T adj(AA^T)b}{\det(AA^T)} \tag{A.3}$$

where a_i is the ith column of matrix A. Let us first try to prove that the above equation is true for $i = 1$. Comparing Eqn.A.1 with Eqn.A.3, we see that it is sufficient to prove

$$a_1^T adj(AA^T)b = \det(A_1^* A^T) - det(A_1^o {A_1}^{oT}) \tag{A.4}$$

Now we replace all the determinants with the formula

$$\det(A_{n\times n}) = \Sigma Sgn(j_1, j_2, j_3, \cdots, j_n) a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}$$

The summation is over all possible permutations of $j_1, j_2, j_3, ..., j_n$ Where

$$Sgn(j_1, j_2, j_3, ..., j_n) = 1$$

if the number of times a larger index precedes a smaller index is even

$$Sgn(j_1, j_2, j_3, ..., j_n) = -1$$

if the number of times a larger index precedes a smaller index is odd
Using this formula on the RHS of Eqn.A.4, we get

$$
\begin{aligned}
RHS \\
&= \sum Sgn(j_1,\cdots,j_m)(b_1 a_{j_1 1} + \sum_{i=2}^{n} a_{1i}a_{j_1 i})(b_2 a_{j_2 1} + \sum_{i=2}^{n} a_{2i}a_{j_2 i}) \\
&\qquad \cdots (b_m a_{j_m 1} + \sum_{i=2}^{n} a_{mi}a_{j_m i}) \\
&- \sum Sgn(j_1,\cdots,j_m)(\sum_{i=2}^{n} a_{1i}a_{j_1 i})(\sum_{i=2}^{n} a_{2i}a_{j_2 i})\cdots(\sum_{i=2}^{n} a_{mi}a_{j_m i}) \\
&= \sum Sgn(j_1,\cdots,j_m)(a_{j_1 1}[b_1 - a_{11}] + \sum_{i=1}^{n} a_{1i}a_{j_1 i}) \\
&\quad (a_{j_2 1}[b_2 - a_{21}] + \sum_{i=1}^{n} a_{2i}a_{j_2 i})\cdots(a_{j_m 1}[b_m - a_{m1}] + \sum_{i=1}^{n} a_{mi}a_{j_m i}) \\
&- \sum Sgn(j_1,\cdots,j_m)(\sum_{i=1}^{n} a_{1i}a_{j_1 i} - a_{11}a_{j_1 1})(\sum_{i=1}^{n} a_{2i}a_{j_2 i} - a_{21}a_{j_2 1}) \\
&\qquad \cdots (\sum_{i=1}^{n} a_{mi}a_{j_m i} - a_{m1}a_{j_m 1})
\end{aligned}
\tag{A.5}
$$

Let us name the two summation terms in Eqn.A.5 as

$$
\begin{aligned}
S_1 &= \sum Sgn(j_1,\cdots,j_m)(b_1 a_{j_1 1} + \sum_{i=2}^{n} a_{1i}a_{j_1 i})(b_2 a_{j_2 1} + \sum_{i=2}^{n} a_{2i}a_{j_2 i}) \\
&\qquad \cdots (b_m a_{j_m 1} + \sum_{i=2}^{n} a_{mi}a_{j_m i}) \\
S_2 &= \sum Sgn(j_1,\cdots,j_m)(\sum_{i=2}^{n} a_{1i}a_{j_1 i})(\sum_{i=2}^{n} a_{2i}a_{j_2 i})\cdots(\sum_{i=2}^{n} a_{mi}a_{j_m i})
\end{aligned}
\tag{A.6}
$$

Therefore

$$RHS = S_1 - S_2 \tag{A.7}$$

All the terms in S_2 can be classified into four types

Type I:

$$\sum Sgn(j_1,\cdots,j_m) \sum_{P(l_k)} (-1)^p (a_{l_1 1}a_{j_{l_1} 1})(a_{l_2 1}a_{j_{l_2} 1})\cdots(a_{l_p}a_{j_{l_p} 1})(\sum_{i=1}^{n} a_{l_{p+1},i}a_{j_{l_{p+1}} i}\cdots\sum_{i=1}^{n} a_{mi}a_{j_m i})$$

for $p = 2, 3, \cdots, n$. Where, $\sum_{P(j_i)}$ stands for summation over all permutations of l_k.

$l_k = 1, 2, \cdots, m$

Now changing the order of summations, we get

$$\sum_{P(l_k)} (-1)^p \sum Sgn(j_1, \cdots, j_m)(a_{l_1 1} a_{j_{i_1} 1})(a_{l_2 1} a_{j_{i_2} 1}) \cdots (a_{l_p} a_{j_{i_p} 1})(\sum_{i=1}^{n} a_{l_{p+1}, i} a_{j_{i_{p+1}} i} \cdots \sum_{i=1}^{n} a_{mi} a_{j_m i})$$

Let us first consider the inner summation. For any given term in the inner summation (*i.e.*, for any permutation of $j_i s$), we can always find some j_u and j_v such that when their positions are exchanged, we get the same term but with an opposite sign. Since the total number of permutations of j_i is always even, all the terms in the inner summation get cancelled. Therefore, the inner summation is zero, which means that the above term is identically equal to zero.

Type II :

$$\sum Sgn(j_1, \cdots, j_m)(-1)(a_p a_{j_p 1})(\sum_{i=1}^{n} a_{1i} a_{j_2 i} \cdots \sum_{i=1}^{n} a_{p-1,i} a_{j_{p-1} i} \sum_{i=1}^{n} a_{p+1,i} a_{j_{p+1} i} \cdots \sum_{i=1}^{n} a_{mi} a_{j_m i})$$

For these terms we can not use the above technique and so they don't get cancelled. This term actually gets cancelled by a similar term in S_1.

Type III :

$$\sum Sgn(j_1, \cdots, j_n)(-1)^m a_{11} a_{j_1 1} a_{21} a_{j_2 1} \cdots a_{m1} a_{j_m 1}$$

This term is similar to the term in Type I and we can always find some j_u and j_v such that when their positions are exchanged, we get the same term but with an opposite sign. Therefore this term is identically equal to zero.

Type IV:

$$\sum Sgn(j_1, j_2, j_3, \cdots, j_n)(\sum_{i=1}^{n} a_{1i} a_{j_1 i})(\sum_{i=1}^{n} a_{2i} a_{j_2 i}) \ldots\ldots (\sum_{i=1}^{n} a_{mi} a_{j_m i})$$

This term too gets cancelled by a similar term in S_1

After cancelling the above terms we will be left with the following terms in S_1

$$\sum_{k=1}^{m}\sum Sgn(j_1,\cdots,j_m)a_{j_k1}b_k\prod_{l\neq k}\sum_{i=1}^{n}a_{li}a_{j_li} \tag{A.8}$$

$$=\sum_{k=1}^{m}b_k\sum Sgn(j_1,\cdots,j_m)a_{j_k1}\prod_{l\neq k}\sum_{i=1}^{n}a_{li}a_{j_li}$$

$$=\sum_{k=1}^{m}b_k\sum_{p=1}^{m}a_{p1}(-1)^{k+p}\sum_{j_i\neq p}Sgn(j_1,\cdots,j_{m-1})\prod_{l\neq k}\sum_{i=1}^{n}a_{li}a_{j_li}$$

$$={a_1}^T adj(AA^T)b$$

We can similarly get the other elements in the vector X ($i.e$ for $i=2,\cdots,n$).

□

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